COMPUTABLE STRUCTURES AND OPERATIONS ON THE SPACE OF CONTINUOUS FUNCTIONS

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Abstract. We use ideas and machinery of effective algebra to investigate computable structures on the space $C[0,1]$ of continuous functions on the unit interval. We show that $(C[0,1], sup)$ possesses infinitely many computable structures non-equivalent up to a computable isometry. We also investigate if the usual operations on $C[0,1]$ are necessarily computable in every computable structure on $C[0,1]$. Among other results, we show that there is a computable structure on $C[0,1]$ which computes $+$ and the scalar multiplication, but does not compute the operation of pointwise multiplication of functions. Another unexpected result is that there exists more than one computable structure making $C[0,1]$ a computable Banach algebra. All our results have implications to the study of the number of computable structures on $C[0,1]$ in various commonly used signatures.

Contents

1. Introduction

In 1930's, Turing, Kleene, Markov and others gave different but actually equivalent formal definitions of what is meant by an effective procedure. Remarkably, Turing immediately tested his new formal approach in analysis. In his early papers [32, 33], Turing gave a formal definition of a computable real. In modern terms, a real r is computable if there is an effective procedure (Turing machine) which, on input s, outputs a rational q such that $|q-r| < 2^{-s}$. Clearly, not every real is computable, because there are only countably many Turing machines.

Turing's definition has a natural generalization to functions. We say that a function $f : [0,1] \to \mathbb{R}$ is computable if there is an effective procedure which, on input s, outputs a tuple of rationals $\langle q_0, \dots, q_n \rangle$ such that $\sup_{x\in[0,1]}\{|f - \sum_{i=0}^{n} q_i x^i|\} < 2^{-s}$. In fact, there are several equivalent ways of saying that a function from reals to reals is computable [6]. We state here some classical and recent results. Myhill [24] showed that there exists a computable function which is differentiable, but does not have a computable derivative. In contrast, Pour-El and Richards [27] showed that if the second derivative of a computable function f exists (but is not necessarily effective), then the derivative of f is computable. Results of this kind belong to a field of mathematics called *computable analysis* [26, 6]. Recent studies have uncovered an unexpected interaction of differentiability and algorithmic randomness (see Nies [25]). For further correlations of differentiability of continuous functions and algorithmic randomness see [3, 4].

We would like to have a notion of computability for other common spaces. Notice that we could use piecewise linear functions with rational breakpoints, or some other *effectively dense subset*, instead of polynomials over $\mathbb{Q}$. In fact, if we have an effectively dense subset of an arbitrary metric space, then we can develop computable analysis on the space:

Definition([6, 26]). Let (M, d) be a complete separable metric space, and let $(q_i)_{i\in\mathbb{N}}$ be a dense sequence without repetitions. The triple $\mathcal{M} = (M, d, (q_i)_{i\in\mathbb{N}})$ is a *computable metric space* if $d(q_i, q_k)$ is a computable real uniformly in i, k. We say that $(q_i)_{i\in\mathbb{N}}$ is a *computable structure* on M.

We refer to the elements of the sequence $(q_i)_{i\in\mathbb{N}}$ as *special points*.

Example. The following metric spaces possess computable structures:

(i) The reals $\mathbb{R}$ with the usual distance metric.
(ii) Cantor space $\{0,1\}^{\mathbb{N}}$, consisting of the functions $f\colon \mathbb{N} \to \{0,1\}$ with the distance function $d(f,g) = \max\{2^{-n}\colon\ f(n) \neq g(n)\}$, (where $\max\emptyset = 0$).
(iii) The space $C[0,1]$ of continuous functions on the unit interval with the pointwise supremum metric.

A *Cauchy name* for a point x is a sequence $(q_{f(s)})_{s\in\mathbb{N}}$ of special points converging to x such that $d(q_{f(s)}, q_{f(t)}) \leq 2^{-s}$ for each $t > s$.

Definition. An element x of a computable metric space $(M, d, (q_i)_{i\in\mathbb{N}})$ is *computable* if there exists a computable function f such that $(q_{f(s)})_{s\in\mathbb{N}}$ is a Cauchy name for x.

To emphasize which computable structure on M is considered, we say that x is computable with respect to $(q_i)_{i\in\mathbb{N}}$ (sometimes written w.r.t. $(q_i)_{i\in\mathbb{N}}$).

1.1. **Equivalent and isometric computable structures.** As we will see, separable spaces have many different computable structures, but not all of these structures are *essentially different*. For instance, rational piecewise linear functions and rational polynomials would lead us to the same notion of a computable function. We have arrived to the following question:

Which computable structures can be considered as equal or similar?

Pour-El and Richards [26] were probably the first to give the most general precise definition of "similar" computable structures:

Definition (Pour-El and Richards [26]). Computable structures $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ on a complete separable metric space (M, d) are *equivalent up to a computable isometry*, or *computably isometric*, if there exists a surjective self-isometry ϕ of M and an effectively uniform algorithm which on input i outputs a Cauchy name for $\phi(\alpha_i)$ in $(\beta_i)_{i\in\mathbb{N}}$.

Computable structures $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ from the definition above can be viewed as computable countable metric spaces. The definition says that these spaces are computably isometric if there exists a computable isomorphism from the closure of $(\alpha_i)_{i\in\mathbb{N}}$ onto the closure of $(\beta_i)_{i\in\mathbb{N}}$. The motivation is clear and is typical to effective mathematics: it is natural to study *computable* objets up to *computable* isomorphisms[1]. The same motivation led *effective algebraists* to the notions of computable categoricity and computable dimension of a *countable algebraic structure*. Our intuition is often based on these classical notions of effective algebra. We explain the notions in the subsection below.

1.2. **Computable algebraic structures and effective algebra.** In contrast to computable analysis, the main objects in effective algebra [2, 10, 13] are effectively presented *countable* algebraic structures:

Definition (Mal'cev [21], independently Rabin [28]). A countable algebra M is constructive (computable) if its elements can be numbered by $\mathbb{N}$ so that all operations on M become computable functions on the respective numbers of elements. The numbering of the universe making the operations effective is often called a constructivization or a computable presentation of M.

Examples of constructive algebras are countable groups with solvable word problem, the field $(\mathbb{Q}, +, \times)$, and the countable atomless Boolean algebra. Mal'cev and

[1] There is another non-equivalent approach motivated by numbering theory [9], we will not discuss it here; see [26, 34, 14]. In this approach, computable structures are considered not up to an arbitrary (computable) isometry ϕ but have to agree up to the *fixed* self-isometry of the underlying space, namely up to the identity self-embedding. Following this approach, the space of reals with the standard distance metric has uncountably many non-equivalent computable strucures, while in our approach all these structures will be computably isometric [17, 23, 26].

Rabin realized that constructive algebras should be considered up to computable isomorphisms:

Definition (Mal'cev, Rabin; 1960's). A constructive algebra M is computably categorical (or autostable) if any two constructivizations of it agree up to a computable automorphism of the algebra. The computable dimension of M is the number of constructivisations of M non-equivalent up to a computable automorphism of M.

The definition above says that in *effective* algebra objects should be considered up to *effective* isomorphisms. Notice that the idea is the same as in the definition of computably isometric structures on a metric space due to Pour-El and Richards.

Computable categoricity has been completely described for abelian p-groups [11, 31], linear orders [29], and Boolean algebras [13]. The theory of computable dimension contains many deep and intriguing results [10]. Algebraic structures from many common classes have computable dimension 1 or ∞ [10, 11, 12], but for any $n \in \mathbb{N}$ there exists a structure having computable dimension n (Goncharov [12]). Structures of finite non-trivial (i.e. ≥ 2) computable dimension can be found in several natural classes including two-step nilpotent groups [16].

Examples of algebraic structures having more than one constructivization, up to a computable isomorphism, include the well-ordering $(\omega, <)$ and the vector space $\mathbb{V}$ over Q of countably infinite dimension. It is well-known that if we add the successor relation S to the signature of ω, then $(\omega, <, S)$ becomes computably categorical. More specifically, *if we restrict ourselves to constructivizations which compute S*, it is easy to construct an isomorphism from one constructivisation to another starting from the left-most point. Similarly, if we add predicates $(P_i(x_0, \ldots, x_i))_{i\in\mathbb{N}}$ to the signature of $\mathbb{V}$, where $P_i(x_0, \ldots, x_i) = 1$ iff $x_0, \ldots, x_i$ are linearly independent, then $\mathbb{V}$ becomes computably categorical *in this new signature.* Indeed, it is sufficient to map a basis to a basis stage-by-stage. As we can see, *the number of non-equivalent constructivisations depends on the choice of the signature.*

1.3. **Computably categorical separable spaces.** Interestingly, computable analysis has been developing quite independently from effective algebra. Combining ideas of Pour-El and Richards, Mal'cev, and Rabin, we obtain the following new definition:

Definition ([23]). A separable space is *computably categorical* if it has a unique computable structure, up to computable isometries. The computable dimension of a separable space is the number of computable structures on it, up to computable isometries.

Following the general philosophy of effective mathematics, we ask:

Main Questions. Which common separable spaces are computably categorical? If a space is not computably categorical, what is its computable dimension? How is computable categoricity dependent on signatures?

The fields of effective algebra and computable analysis contain similar ideas, and one would expect that methods of one field can be adjusted to yield similar results in the another. Nonetheless, even adapting basic technical ideas of effective algebra to computable analysis can be quite hard. In effective algebra we could deal with elements of a given structure directly. For example, we could decide if two elements equal or not effectively. In contrast, the equality on a computable separable space does not have to be decidable. For instance, if two computable reals are *not* equal, then we will eventually see it. However, if they *are* equal, then we may never detect it in finite time. Since we have to deal with Cauchy names of points rather than the points themselves, the complexity of usual arguments will tend to increase by one jump. For example, a finite injury argument will likely become an infinite injury argument, unless we do some specific work to simplify it.

The difficulty of the task and the very little interaction of the fields partially explain why, in contrast to effective algebra, not much is known about computably categorical separable spaces. We list below virtually everything that is known. The metric space l_1 is not computably categorical [26], every separable Hilbert space is computable categorical [17, 23, 26, 5]. Cantor space and the Urysohn space are computably categorical [23]. There is also a description of computably categorical compact subsets of $\mathbb{R}^n$ [23], and two rather specific sufficient conditions of computable categoricity [23, 17]. Also, the space $C[0,1]$ of continuous functions is not computably categorical [23]. Nothing has been done so far on the computable dimensions of uncountable spaces.

As we have discussed above, in computable algebra computable categoricity tends to be dependent on the signature. However, very little is known about this effect in uncountable metric spaces. In the following, we will implicitly use the definition of a computable operation on a space, the formal definition will be given in the preliminary section (see Definition 2.1). The notion is technical but natural, and the reader can safely rely on her/his intuitive understanding of this phenomenon in the discussion below.

It is not difficult to show that the operations $+$ and $(r\cdot)_{r\in Q}$ are computable in *every* computable structure on a separable Hilbert space $(H, d, 0)$, where 0 is the distinguished point zero. This fact can be used to show that separable Hilbert spaces are computably categorical [23]. On the other hand, the operation $x \to (1/2)x$ does not have to be computable in every computable structure on $(C[0,1], sup, 0)$, and this implies that $C[0,1]$ is not computably categorical [23] (the implication is not straightforward). More generally, we arrive at:

Problem. Understand the algorithmic properties of the common operations (such as $+$) on classical Banach spaces and Banach algebras.

The problem above is interesting in its own right and has an analogy in effective algebra (see [15] for *degree spectra of relations*). As we have seen, it is also closely related to the Main Questions stated above.

1.4. **The space $C[0,1]$.** We test our notions on the space $C[0,1]$ of continuous functions on the unit interval with the usual pointwise supremum metric. Our choice is not arbitrary. First of all, this space is (classically) very well understood (see, e.g., [8]). Also, as we have already mentioned, there is a tradition of studying effective properties of continuous functions rooted in the works of Turing. Logical aspects of $C[0,1]$ have been studied intensively as well; this is a long tradition probably going back to Polish school of topology (see, e.g., Mazurkiewicz [22]). More recent investigations include results on hierarchies of continuous functions in relation to descriptive set theory and differentiability (see Kechris and Woodin [18]). Further results can be found in, e.g., [20, 19, 1]. See also [35] for more about hierarchies of continuous functions.

It is well-known that $C[0,1]$ is a universal metric space [30]. In fact, Cherlin proved that the first-order theory of $C[0,1]$, in the signature of rings, is not decidable [7]. $C[0,1]$ is a Banach space and a Banach algebra. Thus, we have a plethora of signatures to play with.

As we have mentioned, $(C[0,1], sup)$ is not computably categorical [23]. What is its computable dimension? Will it become computably categorical if we add $+$ and $(r\cdot)_{r\in Q}$ to its signature? If not, how many extra operations should we add to the signature of $C[0,1]$ to make it computably categorical? Which operations on $C[0,1]$ can be effectively reconstructed from the metric and other operations? Our main results answer these questions.

1.5. **Results.** Recall that $0'$ stands for the halting problem. Goncharov (see, e.g., [10]) showed that if a countable algebra $\mathcal{A}$ has two constructivizations which are isomorphic relative to $0'$ but not computably isomorphic, then the computable dimension of $\mathcal{A}$ is infinite. In Theorem 3.4 we adapt this machinery to separable spaces. Theorem 3.4 gives a sufficient condition for a separable space to have infinitely many computable structures non-equivalent up to a computable isometry. Its proof is of some technical interest because it is one of the rare applications of the priority method in classical computable analysis. As a corollary (see Corollary 3.12), we obtain:

Theorem. There exist infinitely many pairwise computably non-isometric computable structures on $(C[0,1], sup)$.

The theorem above is not a straightforward consequence of Theorem 3.4. For instance, in Lemma 3.10 we show that there exists a computable structure on $(C[0,1], sup)$ in which 0 is a computable point, but the operation $x \to (1/2)x$ is not computable. The proof is different form the one in [23] mentioned above, because we need this structure to satisfy some further properties required in Theorem 3.4.

What if we add $+$ and $(r\cdot)_{r\in Q}$ (in particular, $(1/2)\cdot$) to the signature of $(C[0,1], sup)$? Will the space have a unique structure then? The answer is negative:

Theorem. $(C[0,1], sup)$ is not computably categorical in the signature of Banach spaces.

We prove this theorem by constructing a computable structure on $(C[0,1], sup, +, (r\cdot)_{r\in Q})$ having unusual properties (see Theorem 4.2), including the one stated in the theorem above (see Corollary 4.3). Another property is that the pointwise multiplication $\times$ is not computable w.r.t. this structure (see Corollary 4.4). This fact is of an independent interest to us.

What if we add even more symbols to the signature? Let us consider the signature of Banach algebras with symbols for $sup, +, (r\cdot)_{r\in Q}$, as well as the pointwise multiplication of functions $\times$ and the multiplicative identity 1. It is not difficult to see that $C[0,1]$ in the signature of Banach algebras augmented by a distinguished symbol for the linear function $f(x) = x$, is computably categorical. Clearly, every polynomial can be generated from the monomial x using the usual operations of a Banach algebra. Since the monomial is exactly the function $f(x) = x$ added to the signature, we can conclude that all polynomials with rational coefficients form a uniformly computable set. Therefore, we can effectively map any computable structure to another, but only when restricted to *this signature*.

Notice that a lot of classical and effective theory can be developed just based on the signature of Banach algebras, without this extra symbol for $f(x) = x$. It is rather unexpected that *without this symbol for* $f(x) = x$ the space $C[0, 1]$ is not computably categorical:

Theorem. $(C[0, 1], sup)$ is not computably categorical in the signature of Banach algebras.

The strategy for the theorem above would be to construct a computable structure on $(C[0, 1], sup, +, (r\cdot)_{r\in Q}, \times)$ in which $f(x) = x$ is not a computable point with respect to this structure (see Theorem 5.3). The theorem is a consequence of this fact (see Corollary 5.4).

Informally, the theorem above shows that polynomials are essential and intrinsic to the standard effective analysis.

1.6. **The structure of the paper.** In Section 2 we give a necessary background and a careful elementary analysis of common operations on $C[0, 1]$. For instance, we show that the complicated signatures of Banach spaces and Banach algebras on $(C[0, 1], sup)$ can be equivalently replaced by $\langle +\rangle$ and $\langle +, \times\rangle$, respectively. Section 3 studies the computable dimension of $C[0, 1]$. In Section 4 we show that $C[0, 1]$ is not computably categorical as a Banach space, and in Section 5 we prove that there is more than one structure which makes $C[0, 1]$ a computable Banach algebra.

2. Preliminaries

We give formal definitions of the notions informally used in the introduction. Most of these facts, maybe in a slightly different terminology, can be found in [26, 6].

2.1. **Notations and conventions.** Recall that, given a computable structure $(q_i)_{i\in\mathbb{N}}$ on a metric space M, an element x of M is *computable* if there exists a computable function f such that $(q_{f(s)})_{s\in\mathbb{N}}$ is a Cauchy name for x. It is well-known that a point x from $\mathcal{M} = (M, d, (q_i)_{i\in\mathbb{N}})$ is computable if, and only if, from a positive rational δ one can compute p such that $d(x, q_p) \leq \delta$. We will use this fact without explicit reference. Recall also that, to emphasize which computable structure on M is considered, we say that x is computable with respect to $(q_i)_{i\in\mathbb{N}}$ (sometimes written w.r.t. $(q_i)_{i\in\mathbb{N}}$).

We usually identify a special point α_i with its number i and say "find a special point such that ..." instead of "find a number i such that α_i ...".

Definition 2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be computable metric spaces. A map $F\colon \mathcal{M} \to \mathcal{N}$ is *computable* if there is a Turing functional Φ such that, for each x in the domain of F and for every Cauchy name χ for x, the functional Φ enumerates a Cauchy name for $F(x)$ using χ as an oracle[2].

To emphasize which computable structures we consider, we say that a map F is computable with respect to $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ (written w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$). The composition of two computable maps is computable.

In the special case of isometric (more generally, bi-Lipschitz) maps, Definition 2.1 is equivalent to saying that for every special point α_i in $\mathcal{M}$ the point $F(\alpha_i)$ is computable uniformly in i. We will use this observation without explicit reference. For instance, our definition of computably isometric structures that we used in the introduction, is equivalent to:

[2]That is, $(\Phi^\chi(n))_{n\in\mathbb{N}}$ is a Cauchy name for $F(x)$.

Definition 2.2. Computable structures $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ on a Polish space (M,d) are said to be *equivalent up to a computable isometry* or *(computably) isometric*, if there exists a surjective self-isometry U computable w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$.

Note that if U is a computable surjective isometry, then U^{-1} is computable as well. Therefore, equivalence up to a computable isometry is an equivalence relation on computable metric spaces.

2.2. **Computable operations on spaces.** We follow [23] in our terminology. An operation is a function which maps tuples of points to points (such as the addition in a Banach space), or tuples of points to reals (such as the inner product in a Hilbert space). Also, we view a distinguished point x as function $T_x : M \to \{x\}$ such that $T_x(y) = x$, for every y. Thus, distinguished points are operations of a special kind.

In the following, we view a direct power M^k of (M,d) as a metric space with the metric $d_k = \sup_{i\leq k} d(\pi_i x, \pi_i y)$, where π_i is the projection on the i-th component. Let $(\alpha_i)_{i\in\mathbb{N}}$ be a computable structure on (M,d). The computable structure $[(\alpha_i)_{i\in\mathbb{N}}]^k$ on (M^k, d_k) is the effective listing of k-tuples of special points from $(\alpha_i)_{i\in\mathbb{N}}$.

For convenience, if an operation $X : M^k \to M$ is computable w.r.t. $[(\alpha_i)_{i\in\mathbb{N}}]^k$ and $(\alpha_i)_{i\in\mathbb{N}}$, we simply say that X is computable w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$. Similarly, instead of saying that an operation $X : M^k \to \mathbb{R}$ is computable w.r.t. $[(\alpha_i)_{i\in\mathbb{N}}]^k$ and $(q_i)_{i\in\mathbb{N}}$, where $(q_i)_{i\in\mathbb{N}}$ is the usual effective listing of rationals, we say that X is computable w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$.

Recall that every Turing functional Φ_e can be effectively identified with its computable index e. Thus, we may also speak of uniformly computable families of maps between computable metric spaces.

Definition 2.3. Let $(M, d, (X_j)_{j\in J})$ be a metric space with distinguished operations $(X_j)_{j\in J}$, where J is a computable set. We say that $(\alpha_i)_{i\in\mathbb{N}}$ is a *computable structure* on $(M, d, (X_j)_{j\in J})$ if $(M, d, (\alpha_i)_{i\in\mathbb{N}})$ is a computable metric space and the operations $(X_j)_{j\in J}$ are computable w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$ uniformly in their respective indices $j \in J$.

Example 2.4.

(1) A dense set $(\alpha_i)_{i\in\mathbb{N}}$ is a computable structure on a Banach space $\mathbb{B}$ if $(B, d, (\alpha_i)_{i\in\mathbb{N}})$ is a computable metric space and 0, $+$, and $(r\cdot)_{r\in Q}$ are uniformly computable operations w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$.

(2) Similarly, a collection of points $(\alpha_i)_{i\in\mathbb{N}}$ is a computable structure on a Banach algebra $\mathbb{B}$ if the Banach space operations from (1) above and, additionally, the operation $\times$ and the identity function $1 \in C[0,1]$ are computable w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$.

Clearly, an isomorphism U of a space M_1 onto M_2, in the signature augmented by $(X_j)_{j\in J}$, should *respect* the operations $(X_j)_{j\in J}$.

Definition 2.5. A space $(M, d, (X_j)_{j\in J})$ is *computably categorical* if every two computable structures $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ on $(M, d, (X_j)_{j\in J})$ are computably isometric via an isometry which respects X_j for every $j \in J$.

To emphasize which signature we consider, we frequently use the following terminology:

Convention 2.6. We always assume that we have a metric, and we often omit the symbol for the metric in the signature. (This convention is similar to omitting the equality in model theory.) We say that a metric space is computably categorical as a Banach space if it is considered in the signature $\langle 0, +, (r\cdot)_{r\in Q}\rangle$ of Banach spaces. We say that it is computably categorical as a Banach algebra if we use the signature $\langle 0, +, (r\cdot)_{r\in Q}, 1, \times\rangle$ of Banach algebras.

Definition 2.7. We say that operations $(Y_i)_{i\in I}$ *effectively determine* operations $(X)_{j\in J}$ on a metric space (M, d) if, every isometry of M which respects $(Y_i)_{i\in I}$ respects $(X_j)_{j\in J}$ as well and, furthermore, for any given structure $(\alpha_i)_{i\in\mathbb{N}}$ on (M, d), the uniform computability of $(Y_i)_{i\in I}$ w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$ implies the uniform computability of $(X)_{j\in J}$ w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$.

In the definition above we could have omitted the part talking about isometries respecting operations, but we wish to emphasize that we are not computing *some* image of $(X)_{j\in J}$, but there is only one image of $(X)_{j\in J}$ which we can furthermore compute.

Fact 2.8. Suppose $(M, d, (X_j)_{j\in J}, (Y_i)_{i\in I})$ is computably categorical, where the operations $(Y_i)_{i\in I}$ effectively determine the operations $(X_j)_{j\in J}$. Then $(M, d, (Y_i)_{i\in I})$ is computably categorical.

Proof. Follows at once from Definition 2.5 and Definition 2.7. □

The fact above says that, if a certain operation is determined by other operations, it can be omitted from the signature without any effect on the effective properties of the space.

2.3. **Operations on Banach spaces.** Mazur and Ulam (see, e.g., [30]) showed that every isometry in a Banach space is affine. We will use an effective version of this classical result:

Fact 2.9. Let B be a computable Banach space with computable structure $(\alpha_i)_{i\in\mathbb{N}}$. If $(\beta_i)_{i\in\mathbb{N}}$ is a computable structure on $(B, d, 0)$, and $(\beta_i)_{i\in\mathbb{N}}$ computably isometric to $(\alpha_i)_{i\in\mathbb{N}}$, then $+$ and $(r\cdot)_{r\in Q}$ are uniformly computable w.r.t. $(\beta_i)_{i\in\mathbb{N}}$.

Here note that a computable isometry does not have to preserve anything except for the metric.

Proof. See the proof of [23, Fact 3.6]. □

In the following, $+$ stands for the pointwise addition of functions, and $\times$ denotes the pointwise multiplication of functions. We begin with some easy facts about computable structures on metric and Banach spaces[3].

Fact 2.10. *In a computably separable Banach space the operations $+$ and $d(\cdot,\cdot)$ effectively determine the operations $(r\cdot)_{r\in Q}$, $-$ and the zero element $\mathbf{0}$.*

[3]The authors thank Diamondstone for pointing out Facts 2.10, 2.11 and 2.15.

Proof. Fix a computable structure B on a Banach space where we are only given $+$ as computable operation on B. In a Banach space the zero element $\mathbf{0}$ is defined uniquely by the formula $d(x, x+x) = 0$. Given n we search for some $b_n \in B$ such that $d(b_n, b_n + b_n) < 2^{-n-1}$. Then $\{b_n\}_{n\in\mathbb{N}}$ is a fast converging sequence with limit $\mathbf{0}$.

To show that we can effectively determine the operation $-$, it is sufficient to show that given $b_0, b_1 \in B$ we can effectively get a fast converging sequence $\{c_n\}_{n\in\mathbb{N}}$ with limit $b_0 - b_1$, where $c_n \in B$ for all n. We can compute a Cauchy name for $-b_1$, since the latter is uniquely defined by the formula $b_1 + x = \mathbf{0}$. Now it is straightforward to fcompute a Cauchy sequence rapidly converging to $b_0 + (-b_1)$.

Given $n \in \mathbb{N}$ and $b \in B$, the element $\frac{b}{n}$ is uniquely defined by the formula $\underbrace{x + x + \cdots + x}_{n \text{ times}} = b$. Thus given $n, m \in \mathbb{N} - \{0\}$ and $b \in B$ we can compute a Cauchy name for $\frac{m}{n} \cdot b$. □

Fact 2.11. *In a computably separable Banach space the operations $-$ and $d(\cdot,\cdot)$ effectively determine the operations $(r\cdot)_{r\in Q}$ and $+$.*

Proof. Given $-$ we can compute $\mathbf{0}$ using $b - b$ for any $b \in B$. The operation $+$ is then computed using $b_0 + b_1 = b_0 - (\mathbf{0} - b_1)$. Once we have $+$ and the metric $d(\cdot,\cdot)$ we can then compute $(r\cdot)_{r\in Q}$ as above. □

Remark 2.12. From the facts above we conclude that, for our purposes, the signature of Banach spaces can be replaced by either $\langle + \rangle$ or $\langle - \rangle$, the latter two being equivalent.

We will not touch the question below:

Question 2.13. *In a computably separable Banach space do the operations $(r\cdot)_{r\in Q}$ and $d(\cdot,\cdot)$ effectively determine the operation $+$?*

We now turn to Banach algebras. The multiplicative identity in a Banach algebra is denoted as $\mathbf{1}$. In $C[0,1]$ this is defined by the rule $\mathbf{1}(x) = 1$ for each $x \in [0,1]$.

Convention 2.14. In the following, when we consider $C[0,1]$, we write d for the supremum metric.

In fact, it is known that a Banach space automorphism of $C[0,1]$ is already a Banach algebra automorphism if it maps $\mathbf{1}$ to $\mathbf{1}$ (see, e.g., [8]). We need more:

Fact 2.15. *In $C[0,1]$ the operations $\times$, $+$ and $d(\cdot,\cdot)$ effectively determine the elements $\mathbf{0}$ and $\mathbf{1}$, and the operation $(r\cdot)_{r\in Q}$.*

Proof. By Fact 2.10 we can effectively obtain $\mathbf{0}$ and $(r\cdot)_{r\in Q}$. In the standard structure consisting of rational polynomials on $C[0,1]$ (and hence in every isometric structure), the set $\{\mathbf{0}, \mathbf{1}\}$ is isolated by the formula $x \times x - x = \mathbf{0}$, since for every function f satisfying this formula, $f(x) = 0$ or 1 at each $x \in [0,1]$. Given a structure Y on $C[0,1]$ and a special point $a \in Y$ we can compute $a \times a - a$ up to any degree of accuracy. To show that $\mathbf{1}$ is a computable point of Y we search, for any given e, a special point $c_e \in Y$ such that $\|c_e \times c_e - c_e\| < 2^{-e}$ and $|\|c_e\| - 1| < 2^{-e}$. This search is effective since the norm $\|\cdot\|$ is computable, and is easy to check that this procedure will return a special point c_e such that $d(c_e, \mathbf{1}) < 2^{-e}$. □

Remark 2.16. The signature of Banach algebras can be equivalently replaced by $\langle +, \times \rangle$ when considering $C[0,1]$.

The reader may notice that the fact above holds for a large class of computable Banach algebras, but it is not important for us. There are several signatures and further questions which we leave untouched since they seem less natural. For instance: Does $\mathbf{1}$ effectively determine the operation $(1/2)\cdot$ (in presence of the metric)? We believe that the proof of Theorem 3.10 can be modified to answer this question in negative.

Conclusion. The main signatures of interest for $(C[0,1],d)$ are $\langle + \rangle$ and $\langle +, \times \rangle$, the former being equivalent to the signature of Banach spaces, and the latter to the signature of Banach algebras.

3. Limit equivalent computable structures

We now investigate the analogue of a classical result of Goncharov. The main definition is the following.

Definition 3.1. Two computable structures $\mathcal{L}$ and $\mathcal{L}'$ on a separable metric space (M,d) are said to be *limit equivalent* if there is a total computable function $g(x,s) : \mathcal{L} \times \mathbb{N} \mapsto \mathcal{L}'$ of two arguments such that $f(x) = \lim_{s\to\infty} g(x,s)$ is an isometric bijection of $\mathcal{L}$ onto $\mathcal{L}'$, where the limit is taken with respect to the standard metric on $\mathbb{N}$ (i.e., the sequence $(g(x,s))_{s\in\mathbb{N}}$ is eventually stable on every x).

Notice that we require for the number of changes in $g(x,0), g(x,1), g(x,2), \dots$ to be finite for every x. Thus, the function $f(x) = \lim_s g(s,x)$ induces a self-isometry of (M,d) onto itself under which the image of every special point from $\mathcal{L}$ is a special point in $\mathcal{L}'$. Notice that $(d(g(x,s), g(x,s+1)))_{s\in\mathbb{N}}$ does not have to be rapidly converging. Consequently, $f(x)$ does not have to be equal to a computable isometry with respect to $\mathcal{L}$ and $\mathcal{L}'$. We prefer to write $g_s(x)$ instead of $g(x,s)$.

In the following, we identify an element v_n from a computable structure $(v_n)_{n\in\mathbb{N}}$ on a space with the number n. Under this identification, the function f from Definition 3.1 can be viewed as a Δ^0_2 permutation of natural numbers with a special property.

Definition 3.2. A computable structure $\mathcal{L}$ on a separable metric space (M,d) is *rational-valued* if $d(x,y) \in \mathbb{Q}$ for every $x,y \in \mathcal{L}$, and the distance d is represented by a computable *function* of two arguments mapping each pair of special points (x,y) to the corresponding rational number $d(x,y)$.

Note that every rational-valued computable structure $\mathcal{L}$ can be viewed as a computable countable relational model $\langle \mathbb{N}, (D_r)_{r\in\mathbb{Q}} \rangle$, where for each $r \in \mathbb{Q}$ and $x,y \in M$ we have $D_r(x,y) = 1$ if, and only if, $d(x,y) = r$.

Remark 3.3. Not every computable structure on a separable metric space is computably isometric to a rational-valued computable structure. In fact, there are computable spaces which do not have rational-valued dense subsets at all. The simplest Polish space with this property is the Cantor space with the usual ultrametric $d(f,g) = \max\{2^{-n} \colon\ f(n) \neq g(n)\}$, replaced by $d_1(f,g) = \sqrt{2}d(f,g)$.

The main result of the section is:

Theorem 3.4. *Suppose $\mathcal{L}$ and $\mathcal{L}'$ are two computable rational-valued structures on a separable metric space (M,d) which are not computably isometric. If $\mathcal{L}$ and $\mathcal{L}'$*

are limit equivalent, then (M, d) has infinitely many computable structures which are pairwise non computably isometric[4].

Proof. The proof of Theorem 3.4 is organized as follows. First, we state the notations and the requirements. Next, we give an informal description which is followed by the formal construction and its verification.

3.1. **Notations and conventions.** We fix an effective listing $(\Psi_e)_{e\in\mathbb{N}}$ of all partial computable functions of two arguments, which includes all computable isometries from $\mathcal{L}$ to $cl(\mathcal{L}')$. Here for each $S \subseteq M$, $cl(S)$ stands for the completion of S in M. For every x and n such that $\Psi_e(x,n)\downarrow$, the number $\Psi_e(x,n)$ will be interpreted as an element of $\mathcal{L}'$. The listing $(\Psi_e)_{e\in\mathbb{N}}$ satisfies the following conditions:

(1) for every e, t, x, we have $d(\Psi_e(x,t), \Psi_e(x,t+1)) < 2^{-t-1}$, if $\Psi_e(x,t)$ and $\Psi_e(x,t+1)$ converge,
(2) for every stage s and every e, t, x, we have $\Psi_{e,s}(x,t)\downarrow$ only if $\Psi_{e,s}(x,n)\downarrow$ for each $n \leq t$, and
(3) if $\Theta : \mathcal{L} \mapsto cl(\mathcal{L}')$ is a computable isometry then there exists some e such that for every $x \in \mathcal{L}$ we have $\Theta(x) = \lim_{n\to\infty} \Psi_e(x,n)$.

To see that $(\Psi_e)_{e\in\mathbb{N}}$ exists, we start with some universal listing of all partial computable functions of two variables, and limit ourselves to only those which satisfy (1) to (3). Since $d(\Psi_e(x,t), \Psi_e(x,t+1))$ is a computable fast converging sequence of rational numbers (in this case $d(\Psi_e(x,t), \Psi_e(x,t+1))$ is in fact rational), we will always be able to tell whenever $d(\Psi_e(x,t), \Psi_e(x,t+1)) < 2^{-t-1}$.

For every e and x, set $\Theta_e(x) = \lim_{n\to\infty} \Psi_e(x,n)$ if the limit exists (where the limit is taken with respect to the metric on M), and set $\Theta_e(x)\uparrow$ otherwise. The range of Θ_e, of course, does not have to be included in $\mathcal{L}'$.

Notation 3.5. At stage s we set $\Theta_{e,s}(x)$ equal to $\Psi_{e,s}(x,m)$ if m is the largest such that $\Psi_{e,s}(x,m)\downarrow$, and we set $\Theta_{e,s}(x)$ undefined, otherwise. In the former case we let $\theta_{e,s}(x) = m$. Thus, $\Theta_{e,s}(x)$ is our stage s guess about $\Theta_e(x)$, and $\theta_{e,s}(x)$ indicates the error between $\Theta_{e,s}(x)$ and $\Theta_e(x)$.

Let $f = \lim_s g_s$ be a Δ^0_2 permutation of natural numbers witnessing the limit equivalence of $\mathcal{L}$ and $\mathcal{L}'$. As we have mentioned above, $\mathcal{L}$ and $\mathcal{L}'$ are essentially countable models. Thus we can safely assume that g_s is an isometry when restricted to the first s elements of its domain.

Note that if the assumption of being rational-valued was removed then we can no longer assume that $g_s\restriction_s$ is an isometry of finite metric spaces. At each stage s we only see $g_s\restriction_s$ as an isometry "with an an error of at most ε" for some $\varepsilon > 0$. In this more general setting we do not know if any reasonable analogue of Goncharov's theorem holds.

3.2. **Requirements.** We are going to produce a countably infinite family $\{A_m : m \in \mathbb{N}\}$ of computable structures on (M, d) which are pairwise not computably isometric. For every m, the structure A_m will be rational-valued.

[4]As we mentioned above, $\mathcal{L}$ and $\mathcal{L}'$ can be viewed as computable structures that are isomorphic relative to $0'$ but not computably isomorphic. By the Goncherov's theorem there are infinitely many computable versions that are pairwise not computably isomorphic. However, these copies may be computably isometric as computable metric spaces, just not by an isometry that takes special points to special points (recall we need to worry about their *completions*). Thus, the original result of Goncharov cannot be applied.

We need to satisfy, for every $n > m$ and e, the following requirements:

$$\mathcal{N}_{e,m,n} : \Theta_e \text{ does not induce an isomorphism from } cl(A_m) \text{ onto } cl(A_n),$$

and

$$\mathcal{R}_m : A_m \text{ is isometric (in fact, limit equivalent) to } \mathcal{L} \text{ and } \mathcal{L}'.$$

To meet $\mathcal{R}_m$ we will construct surjective isometries between computable structures. Thus, A_m will be a rational-valued computable structure isomorphic to $\mathcal{L}$ and $\mathcal{L}'$ as a relational model (recall the discussion after Definition 3.2).

3.3. **Informal description.** We first describe the strategy for $\mathcal{R}_m$. To meet $\mathcal{R}_m$ we construct Δ^0_2 surjective isometric maps $\xi_m : A_m \mapsto \mathcal{L}$ and $\eta_m : A_m \mapsto \mathcal{L}'$. This is done via the approximations $\xi_{m,s}$ and $\eta_{m,s}$ where $\xi_m = \lim_s \xi_{m,s}$ and $\eta_m = \lim_s \eta_{m,s}$. Additionally we ensure that at each stage s, $g_s(\xi_{m,s}) = \eta_{m,s}$ on their domains:

$$\begin{tikzcd} \mathcal{L} \arrow[r, "g_s"] & \mathcal{L}' \\ A_{m,s} \arrow[u, "\xi_{m,s}"] \arrow[ur, "\eta_{m,s}"'] & \end{tikzcd}$$

The main strategy of $\mathcal{R}_m$ is to copy either $\mathcal{L}$ or $\mathcal{L}'$, which is carried out via the surjective isometric maps ξ_m and η_m built by the strategy. The use of two maps rather than a single one will enable us to organize the activity of switching back and forth between copying $\mathcal{L}$ and copying $\mathcal{L}'$ during the construction. Since $g(x)$ may change several times before stabilizing on a value, it may become necessary for us to redefine ξ_m and η_m during the construction in order to maintain the equality illustrated above. To be more specific, suppose at stage s we have defined $\xi_{m,s}(y)$ and $\eta_{m,s}(y)$ such that $g_s(\xi_{m,s}(y)) = \eta_{m,s}(y)$. Suppose now that $g_{s+1}(x) \neq g_s(x)$ where $x = \xi_{m,s}(y)$. To keep the equality we have to do one of two things: either maintain $\xi_{m,s+1}(y) = \xi_{m,s}(y)$ and redefine $\eta_{m,s+1}(y) = g_{s+1}(x)$, or maintain $\eta_{m,s+1}(y) = \eta_{m,s}(y)$ and redefine $\xi_{m,s+1}(y) = z$ where $g_{s+1}(z) = g_s(x)$ (we speed up the approximation for g_s until such a z is found). In the former case we say that $\mathcal{R}_m$ *corrects via* η_m and in the latter case we say that $\mathcal{R}_m$ *corrects via* ξ_m. Each time $\mathcal{R}_m$ needs to correct, it will choose one of the two sides to preserve; this choice will be made so that the highest priority $\mathcal{N}$-requirement with a current restraint larger than x is not injured. Since the approximation to $g_s(x)$ will eventually stabilize, at the end, η_m and ξ_m will be witnesses to the limitwise equivalence of A_m and $\mathcal{L}$, and the limitwise equivalence of A_m and $\mathcal{L}'$, respectively.

An $\mathcal{N}_{e,m,n}$-strategy in isolation will define a computable isometry between $\mathcal{L}$ and $\mathcal{L}'$ using the approximation $\Theta_{e,s} : A_m \to A_n$ and the maps ξ_m and η_n. Recall that $\Theta_e(x)$, if defined, is equal to the limit of the fast converging sequence $(\Psi_e(x,n))_{n\in\mathbb{N}}$ of points in $\mathcal{L}'$. Recall also that for every s, $\Theta_{e,s}$ is a (partial) function from $\mathcal{L}$ to $\mathcal{L}'$, but the range of Θ_e itself may be outside $\mathcal{L}'$.

If Θ_e is defined but does not induce an isometry, we will eventually see it because $\Theta_{e,s}$ will reflect it at some stage s. (This can only happen if for some $x, y \in \mathcal{L}$, we have $d(x,y) \neq d(\Theta_e(x), \Theta_e(y))$.) The slightly more difficult case to handle is if Θ_e is not total, or Θ_e induces an isometry which is not onto. This, however, can be measured in a Π^0_2-way, and so to circumvent this difficulty, we use *expansionary stages* combined with a continuous version of the original Goncharov's preservation strategy, as follows: We call a stage s (e, m, n)-*expansionary*, if $\Theta_{e,s}$ "looks like an isometry from A_m to A_n with a certain precision" on a larger initial segment of its domain, with a better precision than at the previous expansionary stage, and with a further element of $\mathcal{L}'$ covered by a sufficiently small neighborhood of the range of $\Theta_{e,s}$. (The formal definition of an expansionary stage will be given later). We will show that there are infinitely many (e, m, n)-expansionary stages iff Θ_e induces an onto isometry from $cl(A_m)$ to $cl(A_n)$.

We allow the strategy $\mathcal{N}_{e,m,n}$ to act only at (e, m, n)-expansionary stages. At an (e, m, n)-expansionary stage s of the construction, $\mathcal{N}_{e,m,n}$ will define the length of agreement between $\Theta_e(A_m)$ and A_n (this will be formally defined later) and will attempt to preserve $\xi_{m,s}$ on the domain of $\Theta_{e,s}$ and $\eta_{n,s}$ on the range of $\Theta_{e,t}$ for $t \leq s$. It is crucial that at every finite stage the domain and the (approximation to) range are both *finite sets*. If the restraint of this strategy $\mathcal{N}_{e,m,n}$ eventually covers all of A_n and A_m then we would force both ξ_m and η_n to be computable functions. This would allow us to argue that $\mathcal{L}$ and $\mathcal{L}'$ are (contrary to the assumption of the theorem) computably isometric via the composition of ξ_m^{-1}, Θ_e and η_n.

The preservation strategy of $\mathcal{N}_{e,m,n}$ described above potentially conflicts with the $\mathcal{R}_m$-strategy when $g_s(\xi_m(x))$ changes value for x in the domain of of $\Theta_{e,s}$. Similarly, the preservation strategy of $\mathcal{N}_{e,m,n}$ will potentially conflict with the $\mathcal{R}_n$-strategy when $g_t(\xi_n(y))$ changes value for y in the range of $\Theta_{e,t}$. This is illustrated in the diagram below:

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "g_s"] & \mathcal{L}' \\
A_{m,s} \arrow[u, "\xi_{m,s}"] \arrow[ur, "\eta_{m,s}"] \arrow[r, dotted, "\Theta_{e,s}"'] & A_{n,s} \arrow[u, dotted, "\eta_{n,s}"']
\end{tikzcd}$$

To prevent injuring $\mathcal{N}_{e,m,n}$, the $\mathcal{R}_m$-strategy would redefine η_m instead of ξ_m, while the $\mathcal{R}_n$-strategy would redefine ξ_n instead of η_n. In this way the $\mathcal{R}_m$- and $\mathcal{R}_n$-strategies can maintain their equalities while not injuring $\mathcal{N}_{e,m,n}$. Each $\mathcal{N}_{e,m,n}$ eventually has finite restraint, and since the approximation $(g_s)_{s\in\mathbb{N}}$ will eventually settle on each finite subset of $\mathcal{L}$, the overall construction involves only finite injury.

Definition 3.6. (e, m, n)-*Expansionary stages*: Recall that the elements of computable structures are identified with natural numbers. Hence in the following, $\Theta_{e,s} : A_m \to A_n$ is viewed as a map from $\mathbb{N}$ to $\mathbb{N}$. Given any stage s and e, m, n we let s^* be the largest $t < s$ such that t is an (e, m, n)-expansionary stage (set $s^* = 0$ if such a t does not exist).

We say that a stage s is (e, m, n)-*expansionary* if $s = 0$, or

(1) the domain of $\Theta_{e,s}$ contains a longer initial segment of $\mathbb{N}$ since s^*, and for each $x \leq s^*$, $\theta_{e,s}(x) > s^*$;
(2) for every $x, y \leq s^*$, $|d(x, y) - d(\Theta_{e,s}(x), \Theta_{e,s}(y))| \leq 2^{-s^*+1}$;
(3) the 2^{-s^*}-neighborhood of the range of $\Theta_{e,s}$ contains the initial segment of $\mathbb{N}$ of length at least s^*.

Notice that every (e, m, n)-expansionary stage is associated with an initial segment of the domain of $\Theta_{e,s}$ (see (1)) and also with an initial segment of its range (see (3)). We denote these initial segments by $\sigma_{e,m,n,s}$ and $\tau_{e,m,n,s}$, respectively. To reduce cumbersome notation we drop e, m, n from the subscript when the context is clear.

3.4. **Strategies.** We describe the strategies for each requirement.

Strategy for $\mathcal{N}_{e,m,n}$: If stage s is not (e, m, n)-expansionary then the strategy does nothing. Otherwise it sets the following restraints on the maps ξ_m and η_n until the next (e, m, n)-expansionary stage: Preserve the computation of $\xi_m(x)$ for every $x \leq \sigma_{e,m,n,s}$ and the computation of $\eta_n(y)$ for every

$$y \in L_{e,m,n,s} = \{\Theta_{e,t}(z) : z \leq |\sigma_{e,m,n,t}|, t \leq s\}$$

(notice it is a finite set of points).

Strategy for $\mathcal{R}_m$: At stage s of the construction, we define isometric partial maps $\xi_{m,s} : A_{m,s} \to \mathcal{L}$ and $\eta_{m,s} : A_{m,s} \to \mathcal{L}'$. By the choice of $\mathcal{L}$ and $\mathcal{L}'$, we can safely assume that g_s is an isometry when restricted to first s elements of its domain. We also assume that for every y mentioned before in the construction, there is an element x such that $g_s(x) = y$. The $\mathcal{R}_m$-strategy does the following:

(1) *Correction*: For each x such that $\xi_m(x)$ and $\eta_m(x)$ are currently defined, but $g_s(\xi_m(x)) \neq \eta_m(x)$, we correct via either (i) or (ii):
   (i) *Correction via* η_m: Maintain $\xi_m(x)$ and redefine $\eta_m(x) = g_s(\xi_m(x))$.
   (ii) *Correction via* ξ_m: Maintain $\eta_m(x)$ and redefine $\xi_m(x) = z$ where $g_s(z) = \eta_m(x)$.

   For each x where correction has to be done we pick the highest priority $\mathcal{N}$-requirement such that $\xi_m(x)$ or $\eta_m(x)$ is restrained. We correct via η_m if $\mathcal{N}$ wants to restrain ξ_m, otherwise we correct via ξ_m. Initialize all lower priority $\mathcal{N}$-strategies. (If no $\mathcal{N}$-strategy restrains x then we correct via η_m.)
(2) *Extension*: Let k be the least number which is not in the range of ξ_m. Find an element y in A_m such that $\xi_m(y)$ can be set equal k and $\eta_m(y)$ equal to $g_s(k)$ (i.e., we have to ensure that η_m, ξ_m are isometries of finite metric spaces). If such an element does not exist, introduce a new element y_0 to A_m and for each $y \in A_m$, declare the distances $d(y_0, y)$ correspondingly, i.e., set $d(y_0, y) = d(k, \xi_m(y)) = d(g_s(k), \eta_m(y))$. The extension substage is finished[5].

3.5. **Construction.** We fix an effective priority ordering of the $\mathcal{N}$-strategies. The $\mathcal{R}$-strategies are global strategies and are not assigned a priority, and will not be injured during the construction.

At stage 0 of the construction, initialize all $\mathcal{N}$-strategies. At stage s, let the first s many $\mathcal{N}$-strategies act according to their instructions described above. Next let the first s many $\mathcal{R}$-strategies act.

[5]Recall that we assume that g_s is an isometry on first s elements, for every s. Therefore, we can always fix a $k \leq s$ and the corresponding g_s-image of k. The distances will agree, and we can safely set $d(y_0, y) = d(k, \xi_m(y))$. Obviously, since both $\mathcal{L}$ and $\mathcal{L}'$ are subsets of M, and in fact one is a permutation of the other, there is no further tension here. We also note that (due to other strategies acting) A_m may already have many elements outside the domain of ξ_m, and in this case we will not have to introduce new elements to A_m at this particular stage.

3.6. **Verification.** We first show that each $\mathcal{R}_m$ is met, i.e., A_m is limitwise equivalent to $\mathcal{L}$ via ξ_m and to $\mathcal{L}'$ via η_m.

Lemma 3.7. *For every m, the maps $\xi_m = \lim_s \xi_{m,s}$ and $\eta_m = \lim_s \eta_{m,s}$ are well-defined, bijective, and isometric.*

Proof. The strategy for $\mathcal{R}_m$ cannot be injured. Fix an x, and we argue that $\lim_s \xi_{m,s}(x)$ and $\lim_s \eta_{m,s}(x)$ exists. Let $\mathcal{N}$ be the highest priority strategy that at some stage of the construction wants to preserve the computation of either ξ_m or η_m. Suppose $\mathcal{N}$ wishes to preserve the computation of ξ_m, say at some earliest stage s_0 (note that in this case $\mathcal{N}$ will never want to preserve the computation of η_m). The extension step in the construction ensures that when x is first enumerated in the structure A_m, we immediately define $\xi_m(x)$ and $\eta_m(x)$. Since this values are only redefined but never canceled, we have $\xi_{m,s_0}(x) \downarrow$ and $\eta_{m,s_0}(x) \downarrow$. Clearly $\xi_{m,s_0}(x)$ is never again redefined after s_0, since the correction step for x will always respect requirement $\mathcal{N}$ after s_0. Since $g_s(\xi_{m,s_0}(x))$ will be eventually stable, this means that $\eta_{m,s}(x)$ will be eventually stable. If $\mathcal{N}$ wishes to preserve η_m instead we proceed as above, but since g is onto we have that ${g_s}^{-1}(\eta_{m,s_0}(x))$ will be eventually stable.

The correction step ensures that $g_s(\xi_{m,s}(x)) = \eta_{m,s}(x)$ for every x and s. Hence this equality holds for the stable final values as well. Now it is easy to verify that since $g_s \restriction_s$ is an isometry of finite metric spaces for each s, the construction ensures that $\xi_{m,s}$ is an isometry of finite metric spaces at each step of the construction. Clearly ξ_m is injective because it is an isometry. Now the fact that ξ_m is onto follows easily from the fact that the $g_s(y)$ approximation is eventually stable, and by the action in the extension step. Since $\eta_m = g \circ \xi_m$ it follows that η_m is bijective and an isometry. □

Note that the lemma implies at once that all the sets A_m are computable structures on (isomorphic images of) M.

Lemma 3.8. *For every e, m, n, there are infinitely many (e, m, n)-expansionary stages iff Θ_e induces a computable isometry mapping A_m onto A_n.*

Proof. It is straightforward to check that the right to left direction holds. Suppose there are infinitely many (e, m, n)-expansionary stages. In this case condition (1) of Definition 3.6 ensures that for each x, $\Theta_e(x) = \lim_{t\to\infty} \Psi_e(x, t)$ exists. It suffices to check the following:

(i) For any x, y, we have $d(x, y) = \lim_s d(\Theta_{e,s}(x), \Theta_{e,s}(y))$, since the latter is the distance $d(\Theta_e(x), \Theta_e(y))$ in the closure $cl(A_n)$.
(ii) For any y and any i, there exists some x and s such that $\theta_{e,s}(x) > i$ and $d(\Theta_{e,s}(x), y) < 2^{-i}$.

(i) above ensures that Θ_e induces an isometry in the closures, while (ii) ensures that Θ_e maps onto $cl(A_n)$. It is easy to see that (i) and (ii) follow respectively from conditions (2) and (3) of Definition 3.6. □

Lemma 3.9. *For every e, m, n, $\limsup_t |\sigma_{e,m,n,t}| < \infty$ and $\mathcal{N}_{e,m,n}$ is satisfied.*

Proof. We proceed by induction on $\langle e, m, n\rangle$. Suppose the lemma holds for all smaller indices. Hence there is a stage s_0 after which $\mathcal{N} = \mathcal{N}_{e,m,n}$ is never initialized, i.e. never injured by a higher priority requirement. Suppose that $\lim_{t>s_0} |\sigma_{e,m,n,t}| = \infty$. Then by Lemma 3.8, A_m and A_n are computably isometric. Since Θ_e induces an onto map, for each $z \in \omega$ there is a first (e, m, n)-expansionary stage $\hat{s}_z > s_0$ such that $z \in L_{e,m,n,s}$ for every $s \geq \hat{s}_z$. This means that for each x, z, the first definition for $\xi_m(x)$ received after stage s_0 and the first definition for $\eta_n(z)$ received after stage $\hat{s}_z$ are stable and final. Hence ξ_m and η_n are computable functions. By Lemma 3.7 this means that $\mathcal{L}$ is computably isometric to A_m and $\mathcal{L}'$ is computably isometric to A_n, a contradiction. Since $\lim_t |\sigma_{e,m,n,t}| < \infty$, by Lemma 3.8, $\mathcal{N}$ is satisfied. □

The verification is finished, and the theorem is proved. □

We now apply Theorem 3.4 to $C[0, 1]$. For the rest of this paper, d will stand for the pointwise supremum metric on $C[0, 1]$:

$$d(f, g) = \sup_{x\in[0,1]} \{|f(x) - g(x)|\},$$

and $L = (l_i)_{i\in\mathbb{N}}$ will denote an effective sequence of all continuous piecewise linear functions with finitely many rational breakpoints (written rational p.l. functions), without repetitions. Clearly, $(l_i)_{i\in\mathbb{N}}$ is a computable structure which make $C[0, 1]$ a computable Banach space.

Theorem 3.10. *There exists a rational-valued computable structure X on $(C[0, 1], d)$ which is limit equivalent to L, and such that the constant zero function $\mathbf{0}$ is computable w.r.t. X but the operation that takes each function f to $\frac{1}{2}f$ is not computable w.r.t. X.*

Remark 3.11. Strictly speaking we have to be careful with what we mean by "the operation that takes each function f to $\frac{1}{2}f$", because the operation $\frac{1}{2}\cdot$ is not in the signature. In fact, the theorem of Mazur and Ulam ensures that $\phi(\frac{1}{2}f) = \frac{1}{2}\phi(f)$ for every isometry which maps 0 to 0. Thus, the operation can *classically* be added to the signature with no effect on the isometries of the space. We show that this mathematical fact does not hold *effectively*.

We obtain the following important corollary:

Corollary 3.12. *There exists infinitely many computable structures on $(C[0, 1], d)$ which are pairwise not computably isometric.*

Proof. By Theorem 3.4, it is sufficient to prove that there exists two limit equivalent rational-valued computable structures on $(C[0, 1], d)$ which are not computably isometric. The corollary then follows from [23, Fact 3.6] and Theorem 3.10. □

Proof of Theorem 3.10. The proof combines the proof of [23, Theorem 5.2], with an extra requirement to make the structure limit equivalent to L. We build a computable structure $X = (h_i)_{i\in\mathbb{N}}$ on $(\mathcal{C}[0, 1], d)$ which consists of rational p.l. functions, and in which h_0 is the constant zero function. At every stage s of the construction, we introduce an interpretation $h_{i,s}$ of h_i for every $i \leq s$. The interpretation is an element of L. At a later stage t, we may change our interpretation to be another element of L. Thus, in general $h_{i,s} \neq h_{i,t}$, for $s \neq t$. However, at each stage s of the construction and for every $i, j \leq s$, we maintain the equality $d(h_{i,s}, h_{j,s}) = d(h_{i,s+1}, h_{j,s+1})$, which will ensure that the structure X is computable, and isometric to L via $\{\lim_{s\to\infty} h_{i,s}\}$. We also ensure that $h_{0,0} = h_{0,s}$ the constant zero function for every s, which ensures that $\mathbf{0}$ is computable with respect to X. Let $\{\Psi_e\}_{e\in\mathbb{N}}$ and $\{\Theta_e\}_{e\in\mathbb{N}}$ be as in Notation 3.5 (with $(h_i)_{i\in\mathbb{N}}$ instead of $\mathcal{L}'$).

To ensure that the operation $h_i \mapsto \frac{1}{2}h_i$ is not computable, we need to ensure that for each totally defined Θ_e, there is some p such that $\lim_{s\to\infty} h_{\Theta_{e,s}(p)} \neq \frac{1}{2}h_p$, where the limit is, of course, taken in $cl(L)$.

The modification needed to Theorem 5.2 of [23]*:* The "ugly" rational-valued computable structure from [23, Theorem 5.2] is not limit equivalent to L. The technical reason is that the interpretation of a single element of that structure could be changed infinitely often, i.e. $h_{i,s}$ is changed infinitely often. The construction still works because these changes become smaller at later stages, and so the interpretations converge to an element of $cl(L)$, i.e. $\lim_{s\to\infty} h_{i,s}$ exists in $cl(L)$. If we wish to keep the number of changes to each $h_{i,s}$ finite, we need to modify the diagonalization strategy slightly. Suppose we are diagonalizing against Θ_e. We will use a witness h_p which has constant value $16e$ on some small interval I_e (reserved exclusively for this requirement). The basic strategy will wait for $\Theta_e(p)$ to converge with high accuracy. We then adjust h_p on interval I_e by lowering its value $h_p(z)$ by $8e$ for some $z \in I_e$. This will ensure that Θ_e is killed. To ensure that distances are preserved we need to adjust h_m similarly on I_e for every h_m which takes on values larger than $8e$ on I_e. This new modified construction will leave functions with norm $\leq 8e$ untouched after some stage, and ensure that X is limit equivalent to L.

The formal requirements. We need to ensure the following global requirements:

(1) For every i, there is some s_i such that $h_{i,s_i} = h_{i,t}$ for every $t \geq s_i$.
(2) For every i, j and s, we have $d(h_{i,s}, h_{j,s}) = d(h_{i,s+1}, h_{j,s+1})$.
(3) For each m, there is exactly on k such that $\lim_s h_{k,s} = l_m$.

These three requirements clearly imply that X is a computable structure which is limit equivalent to L. We need to satisfy, for every e, the requirements:

$$N_e : \Theta_e \text{ does not represent } h_i \mapsto \tfrac{1}{2}h_i \text{ in } X.$$

In the following, $(I_e)_{e\in\mathbb{N}}$ stands for some effective listing of disjoint computable closed subintervals of $[0, 1]$. We ensure that for each strategy N_e and each h_i, N_e is only allowed to modify h_i on the interval I_e. More specifically, when N_e requests for the interpretation of h_i to be changed at a stage s, we always ensure that $h_{i,s}(z) = h_{i,s+1}(z)$ for every $z \notin I_e$. The requirements N_e all act independently and at most once during the construction.

The detailed strategy for N_e is as follows. It will have it own witness, a rational p.l. function $w_e \in X$. The function w_e, when first defined at stage $2e$, is equal to $16e$ on the interval I_e, is equal to zero at the end-points of $[0, 1]$, and is linear outside I_e.

Let p be such that $w_e = h_{p,2e}$. The strategy N_e does nothing until it sees a computation $\Theta_{e,s}(p)$ where $\theta_{e,s}(p) > e$. If we have

$$\sup_{z\in I_e} \left| \frac{1}{2}h_{p,s}(z) - h_{\Theta_{e,s}(p),s}(z) \right| = \sup_{z\in I_e} \left| h_{\Theta_{e,s}(p),s}(z) - 8e \right| > 2^{-e+1},$$

then the strategy does nothing for the rest of the construction, and we win N_e simply because

$$\sup_{z\in I_e} \left| h_{\Theta_{e,s}(p),s}(z) - f(z) \right| \leq d\left(h_{\Theta_{e,s}(p)}, f\right) < 2^{-e},$$

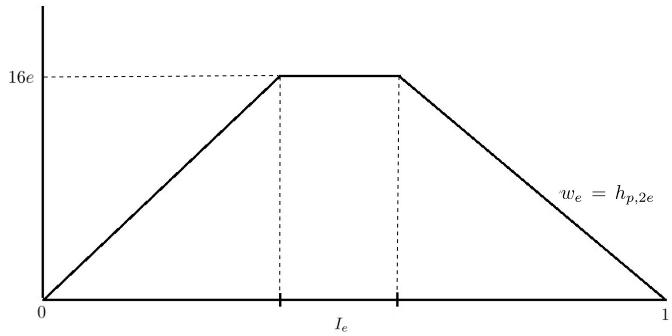

FIGURE 1. Function $h_{p,2e} = w_e$

and thus

$$d\left(\frac{1}{2}h_p(z), f(z)\right) \geq \sup_{z\in I_e}\left|\frac{1}{2}h_p(z) - f(z)\right| = \sup_{z\in I_e}\left|\frac{1}{2}h_{p,s}(z) - f(z)\right| \geq 2^{-e},$$

where $f = \lim_{s\to\infty} h_{\Theta_{e,s}(p)}$. Thus we assume that at stage s we have

$$\sup_{z\in I_e}\left|h_{\Theta_{e,s}(p),s}(z) - 8e\right| \leq 2^{-e+1}. \tag{1}$$

The strategy N_e will then *act* as follows. Introduce a new interpretation $h_{p,t}$ as described below. (Notice that $h_{p,s}$ is equal to $h_{p,2e}$ on the interval I_e, but not necessarily outside this interval.) Choose a (small) sub-interval J of I_e satisfying the following: For all current interpretations $h_{i,s}$ and $h_{j,s}$ of X introduced so far, we have:

(i) $h_{i,s}$ is linear within J, i.e., $h_{i,s}$ has no break-points residing in J.
(ii) There is no pair $z_1, z_2 \in J$ such that $h_{i,s}(z_1) = 8e$ and $h_{i,s}(z_2) \neq 8e$.
(iii) If there is some $z \in J$ such that $h_{i,s}(z) = h_{j,s}(z)$ then $h_{i,s}\restriction_{I_e} = h_{j,s}\restriction_{I_e}$.

It is clear that J can be found effectively, since the construction has only looked at finitely many interpretations so far. Hence each $h_{i,s}$ when restricted to J is either strictly monotonic and does not take value $8e$, or else it is constant on J. Furthermore each pair $h_{i,s}$ and $h_{j,s}$ is either equal or non-intersecting in the interval J.

Now pick z to be the midpoint of J. For every interpretation $h_{i,s}$ such that $h_{i,s}\restriction_{I_e}$ is strictly above $8e$, we set $h_{i,s+1}(z) = 8e$, $h_{i,s+1}(\min J) = h_{i,s}(\min J)$ and $h_{i,s+1}(\max J) = h_{i,s}(\max J)$. We linearly interpolate $h_{i,s+1}$ within J and keep $h_{i,s+1} = h_{i,s}$ unchanged outside J. This is illustrated by Figure 2.

Notice that this action only modifies each $h_{i,s}$ on the interval I_e. It is also straightforward to check the following:

Lemma 3.13. *Distances between the approximations are preserved.*

Proof. Fix i, j. We argue that $d(h_{i,s}, h_{j,s}) = d(h_{i,s+1}, h_{j,s+1})$. Let $m = |h_{i,s}(\min J) - h_{j,s}(\min J)|$ and $M = |h_{i,s}(\max J) - h_{j,s}(\max J)|$. Since there are no breakpoints of $h_{i,s}$ and $h_{j,s}$ in J, we clearly have

$$\sup_{v\in J}|h_{i,s}(v) - h_{j,s}(v)| = \max\{m, M\},$$

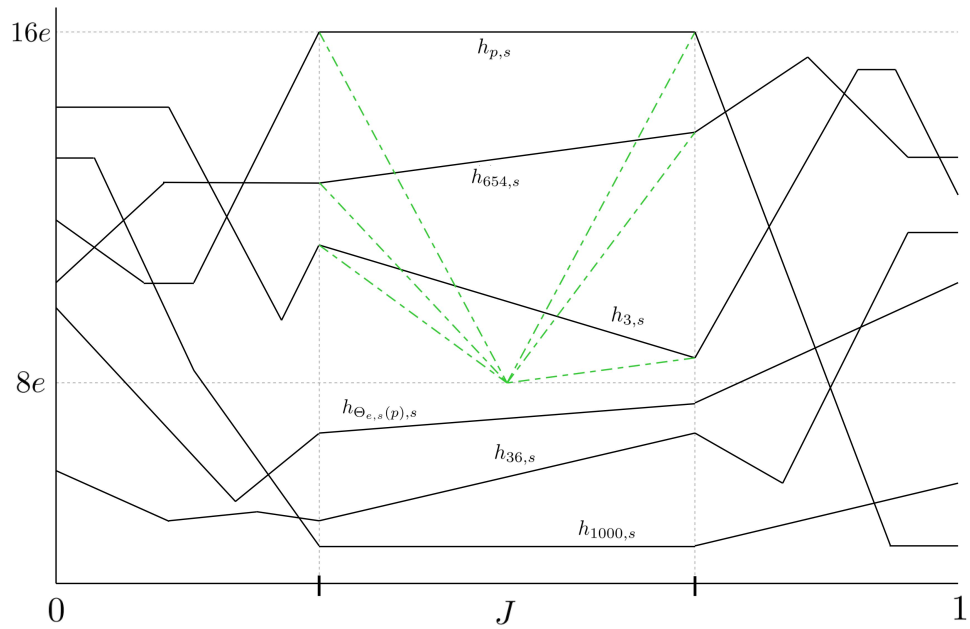

FIGURE 2. The dashed lines indicate the modifications needed to get $h_{i,s+1}$.

therefore, it is sufficient to see that

$$\sup_{v\in J} |h_{i,s+1}(v) - h_{j,s+1}(v)| = \max\{m, M\}.$$

If both $h_{i,s}$ and $h_{j,s}$ are modified then this last equality follows easily from the fact that for every $\min J \le v \le z$, we have $|h_{i,s}(v) - h_{j,s}(v)| \le m$, and for every $z \le v \le \max J$, we have $|h_{i,s}(v) - h_{j,s}(v)| \le M$. Suppose, on the other hand, that $h_{i,s} \neq h_{i,s+1}$ and $h_{j,s} = h_{j,s+1}$. Then for every $v \in J$ we have $h_{i,s}(v) \ge h_{i,s+1}(v) \ge 8e \ge h_{j,s+1}(v) = h_{j,s}(v)$. So we also have that $\sup_{v\in J} |h_{i,s+1}(v) - h_{j,s+1}(v)| = \max\{m, M\}$. □

Lemma 3.14. *N_e is satisfied.*

Proof. If N_e never acts, it is clearly satisfied, so we assume it acts at stage s as above. Since no approximation will ever be changed again within I_e after N_e acts, we have $h_p(z) = h_{p,s+1}(z) = 8e$, and $h_{\Theta_{e,s}(p)}(z) = h_{\Theta_{e,s}(p),s+1}(z) = \min\{8e, h_{\Theta_{e,s}(p),s}(z)\}$. By Equation 1 we have $h_{\Theta_{e,s}(p),s}(z) \ge 8e - 2^{-e+1}$ and so $h_{\Theta_{e,s}(p)}(z) \ge 8e - 2^{-e+1} > 7e$. Now since $\theta_{e,s}(p) > e$ we have $f(z) > 7e - 2^{-e} > 6e > \frac{1}{2}h_p(z)$. Hence $f \neq \frac{1}{2}h_p$. □

Construction. We fix some effective ordering of the N-requirements. At stage s of the *construction*, we simply let strategies of first s requirements act according to their instructions. Next, if we do not see l_m among $(h_{i,s})_{i\le s}$ at stage $s \ge m$, we pick n least such that h_n has no approximation so far and set $h_{n,s} = l_m$. This ends the construction.

Verification. We first show that the global requirements are met. For (1), fix i, and let t be the first stage at which h_i gets its first-ever approximation (namely $h_{i,t}$). Let D be such that $\|h_{i,t}\| = d(0, h_{i,t}) < 8D$. Only the strategies N_e where $e \leq D$ can possibly change the approximation at a later stage. Furthermore, if $t > s$ is such that $h_{i,t} \neq h_{i,s}$, then $\|h_{i,t}\| \leq \|h_{i,s}\|$. Every N-strategy acts at most once. Thus, there is a stage after which h_i will be set to its final value, and so (1) is met. By Lemma 3.13, (2) is met. Finally (3) is met because for each l_m, after a stage where $N_0, \cdots, N_D$ no longer act, where $\|l_m\| < 8D$, any fresh assignment of l_m to a h_i must be stable. Finally, notice that $h_{0,s}$ is never modified during the construction so the interpretation of $\mathbf{0}$ is computable. □

4. $C[0,1]$ is not computably categorical as a Banach space

Recall that the signature of Banach spaces contains $\mathbf{0}$, $+$, and $(r\cdot)_{r\in Q}$. By Fact 2.10 we will assume that the signature only contains $+$, as all the other operations can be reconstructed effectively from $+$. Fact 2.10, Theorem 3.10 or [23, Theorem 5.2] provide us with a corollary which is interesting on its own right:

Corollary 4.1. *There is a computable structure on* $(C[0,1], sup)$ *in which* $+$ *is not computable.*

In the following theorem we show that vector space operations do not effectively determine the multiplicative identity $\mathbf{1}$ in $C[0,1]$. Similarly to Remark 3.11, we need to be careful in our terminology. We will build a computable structure Y on $(C[0,1], d, +)$. Let L stand for the computable structure consisting of rational p.l. functions on the same copy of $(C[0,1], d, +)$. Since we have $cl(L) = cl(Y) = C[0,1]$, every computable Banach space isomorphism ψ of $cl(L)$ onto $cl(Y)$, if existed, would correspond to an automorphism of $(C[0,1], d, +)$. It is well-known[6] that classically every automorphism ψ of $(C[0,1], sup, +)$ is of the form

$$\psi\, f(x) = \delta(x) f(g(x)),$$

where $x \in [0,1]$, $f \in C[0,1]$, the function $\delta(x)$ is either the constant function $\mathbf{1}$ or the constant function $-\mathbf{1}$, and the map g is a homeomorphism of $[0,1]$ onto itself. In fact, in the former case, ψ is a Banach algebra automorphism. Since the automorphism orbit of $\mathbf{1}$ is $\{\mathbf{1},\textbf{-1}\}$, we would have $\psi\mathbf{1} \in \{\mathbf{1},\textbf{-1}\}$. The function $\mathbf{1}$ is clearly a computable point w.r.t. L. Also, notice that $+$ effectively determines $-$, thus $\mathbf{1}$ is computable if, and only if, $\textbf{-1}$ is. Therefore, if we make sure that Y is a computable structure on $(C[0,1], d, +)$ so that $\mathbf{1}$ is not computable w.r.t. Y, then we will have Y is not computably isometric to L (in the signature of Banach spaces).

Theorem 4.2. *There is a computable structure* Y *on* $(C[0,1], d, +)$ *such that* $\mathbf{1}$ *is not a computable point w.r.t. this structure.*

Theorem 4.2 implies:

Corollary 4.3. *The space* $(C[0,1], d, +)$ *is not computably categorical. Equivalently,* $C[0,1]$ *is not computably categorical as a Banach space.*

By Fact 2.15, we have:

[6]See, e.g., Dunford-Schwartz [8] vol. 1, Th. IV.6.26 on page 278.

Corollary 4.4. *There is a computable structure Y on $(C[0,1], d, +)$ such that $\times$ is not a computable operation w.r.t. to this structure.*

We now prove the theorem.

4.1. **Proof idea.** We briefly explain the main intuitive idea behind the proof of the theorem. We will be building a computable structure Y containing $\mathbf{0}$ as a special point. The reader may visualize the idea follows. At every stage of the construction we will have finitely many special points enumerated into Y. At stage s, we think of each point from Y as a rational p.l. function. These will be our current interpretation of Y_s in the usual copy of $C[0,1]$. At a later stage we, however, may be forced to change our current interpretation due to the diagonaliazation requirements. We make $\mathbf{1}$ non-computable w.r.t. the new computable structure we are building. The main diagonaliazation strategy is illustrated on Figure 4. We change the previous interpretation "slightly", but this time preserving $+$ (making sure that if the second element plus the 5'th gave us the 7'th, say, then the same will be true after the "slight" change). Because we will change our interpretation less and less at later stages, *in the limit* the interpretations will converge to some elements of $C[0,1]$ which do not have to be rational p.l. functions. We will make sure that the structure is dense, and the usual operation $+$ on $C[0,1]$ is a computable operation w.r.t. this structure. Consequently, the closures of the standard computable structure and the new one will be (non-computably) isomorphic as Banach spaces via the identity map on $C[0,1]$.

4.2. **Formal proof.** The rest of this section is devoted to the formal proof of Theorem 4.2. The proof has the same flavor as the proof of Theorem 3.10, but the main analytic strategy is different.

Notation 4.5. We fix a dense computable listing of rational p.l. functions $\widehat{L} = (\widehat{l}_n)_{n\in\mathbb{N}}$ on $[0,1]$ satisfying the additional property:

(2) There is a computable sequence of pairwise disjoint closed intervals $\{J_m\}_{m\in\mathbb{N}}$ such that for every m, and every $i \le m$, $\widehat{l}_i$ is constant on J_m.

To arrange for this, we also construct an auxiliary sequence of intervals $\{I_m\}_{m\in\mathbb{N}}$. For each m, we pick an interval $I^*_m \subset I_{m-1}$ such that l_m has no breakpoints in I^*_m, and that $|I^*_m| < \frac{2^{-m}}{C}$, where C is the maximum absolute value of the slope of any linear component of l_m. We also require that if l_m is constant on any subinterval of I_{m-1} then l_m is constant on I^*_m. Now we can modify l_m on I^*_m by the following. Let $a = \min I^*_m$, $b = \max I^*_m$. Set $\widehat{l}_m(z) = l_m(a)$ for every $z \in [a, \frac{a+b}{2}]$ and join the points $(\frac{a+b}{2}, l_m(a))$ and $(b, l_m(b))$ by a straight line. This is illustrated in Figure 3.

Clearly if l_m is constant on I^*_m then $\widehat{l}_m = l_m$. Now let I_m be the left half of I^*_m, and J_{m-1} be any interval disjoint from I_m and $J_{m-1} \subset I_{m-1}$. Since $\{J_m\}$, $\{I_m\}$ and $\{I^*_m\}$ are computable sequences of closed intervals, we have that $\{\widehat{l}_i\}$ is a computable listing of rational p.l. functions (note that we can easily make $\{\widehat{l}_i\}$ a computable listing without repetition). Clearly (2) is satisfied since $\widehat{l}_i$ is constant on I_i which contains J_m as a subinterval for every $m \ge i$.

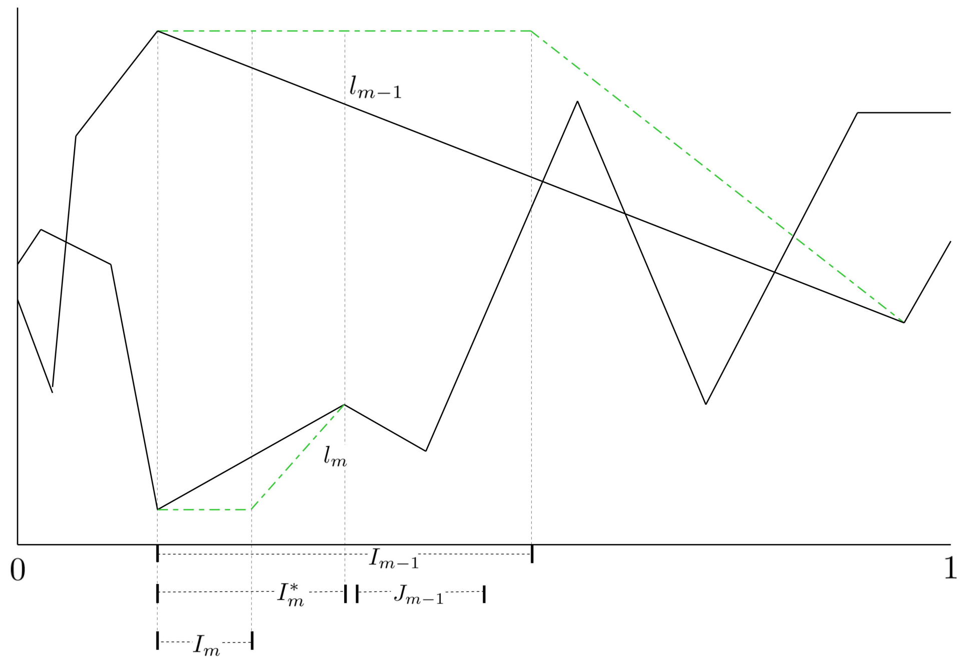

FIGURE 3. The dashed lines indicate the modifications to l_m and l_{m-1}.

Lemma 4.6. (i) *Suppose l_m is constant on some interval I_k. Then $\widehat{l}_m = l_m$.*
(ii) *The set $\widehat{L}$ is effectively closed under addition. That is, there is a computable function h such that given any m_0, m_1, we have $\widehat{l}_{m_0} + \widehat{l}_{m_1} = \widehat{l}_{h(m_0,m_1)}$.*

Proof. (i): Suppose that l_m is constant on I_k. If $k < m$ then l_m is constant on $I_m^* \subset I_k$ and so $\widehat{l}_m = l_m$. If $k \geq m$ then $I_k \subseteq I_{m-1}$ and since l_m is constant on a subinterval of I_{m-1} we would pick I_m^* disjoint from I_k or $I_m^* \subseteq I_k$. The former is impossible.

(ii): Given $\widehat{l}_{m_0}$ and $\widehat{l}_{m_1}$ we can effectively find m such that $l_m = \widehat{l}_{m_0} + \widehat{l}_{m_1}$. Assuming that $I_{m_0} \subseteq I_{m_1}$ we have that l_m is constant on I_{m_0}, so by (i) we can take $h(m_0, m_1) = m$. □

By Lemma 4.6(ii), $\widehat{L}$ is a computable structure on $C[0,1]$ in the signature of Banach spaces. Notice that $\mathbf{1}$ is a special point in $\widehat{L}$.

Lemma 4.7. *The computable structures L and $\widehat{L}$ are computably isometric in the signsture of Banach spaces via the identity map.*

Proof. Clearly for each i, $\sup |l_i(z) - \widehat{l}_i(z)| < 2^{-i}$, by the choice of $|I_i^*|$. It is then straightforward to check that each l_n is a computable point of $\widehat{L}$ (uniformly in n). Hence the identity map from $\widehat{L}$ to L is an onto isometry (in the closure) preserving $+$. □

We will henceforth, in this proof, use $\widehat{L}$ instead of L as the "nice" structure on $C[0,1]$; due to Lemma 4.7, the proof of Corollary 4.3 will still work with $\widehat{L}$ instead of L.

4.3. **Strategies.** We build a computable double sequence of rational p.l. functions $(h_{i,s})_{i,s\in\mathbb{N}}$ from $\widehat{L}$. We define $Y = (h_i)_{i\in\mathbb{N}}$. We ensure that the map $h_i = \lim_s h_{i,s}$ is an isometry taking Y to $\widehat{L}$. To this end we need to maintain the following global requirements:

(1) For every i, $\lim_s h_{i,s}$ exists in $cl(\widehat{L})$.
(2) For every i,j and s, we have $d(h_{i,s}, h_{j,s}) = d(h_{i,s+1}, h_{j,s+1})$.
(3) For every i,j,k and s, we have $h_{i,s} + h_{j,s} = h_{k,s} \Rightarrow h_{i,s+1} + h_{j,s+1} = h_{k,s+1}$.
(4) For each m and each e, there is some k such that $d(\lim_s h_{k,s}, \widehat{l}_m) \leq 2^{-e}$.

It will also be explicite in the construction that Y is effectively closed under $+$. These global requirements ensure that Y is a computable structure on $(C[0,1], d, +)$, and that ϕ is an onto isometry preserving $+$, since ϕ is an onto isometry of finite structures preserving $+$ at each finite stage. For simplicity, we abuse our notations and (notationally) identify h_i with its limiting $\widehat{L}$-interpretation $h_i = \lim_s h_{i,s}$.

Recall Notation 3.5. The main diagonalization requirement is:

$$N_e : \Theta_e(0) \text{ does not represent } \mathbf{1} \text{ in } Y \text{ with respect to } (h_i)_{i\in\mathbb{N}},$$

If we meet N_e for every e, then $\mathbf{1}$ will not be computable in Y.

Strategy for N_e. Wait for a stage s at which $\Theta_{e,s}(0)$ outputs a number x such that

$$d(h_{x,s}, \mathbf{1}) \leq 2^{-e+1}.$$

and $\theta_{e,s}(0) > e$. Notice that at every stage we are dealing with elements of $\widehat{L}$ in which $\mathbf{1}$ *is* a special point of $\widehat{L}$. Thus, the inequality above can be checked effectively at every stage. If we ever see such an s and x, we effectively choose a closed subinterval $J = J_{k_0}$, for some fresh k we have never before used, at which all interpretations we have introduced so far are constant functions (recall Notation 4.5). We also assume k_0 is large enough so that all interpretations are constant on I_{k_0} as well.

Let $h_{0,s}, h_{1,s}, \ldots, h_{n,s}$ be all interpretations we have introduced so far. We now modify $h_{0,s}, h_{1,s}, \ldots, h_{n,s}$ as follows. Let $a = \min J$, $b = \max J$ and z be the midpoint of J. For each i we set $h_{i,s+1}(z) = (1-2^{-e+3})h_{i,s}(a)$, $h_{i,s+1}(a) = h_{i,s}(a)$ and $h_{i,s+1}(b) = h_{i,s}(b)$, and make $h_{i,s+1}$ linear on $J - \{z\}$. We keep $h_{i,s+1} = h_{i,s}$ outside J. This is illustrated in Figure 4:

For each i we can effectively obtain an index m such that $l_m = h_{i,s+1}$. Since l_m is constant on I_{k_0+1} we apply Lemma 4.6(i) to get $\widehat{l}_m = l_m$. We then let $\widehat{l}_m$ to be the new interpretation of h_i.

In this case we say that the strategy N_e *acts*. We will never modify any h_i within $J = J_{k_0}$ again.

Lemma 4.8. *Distances and the operation $+$ are preserved under this action.*

Proof. Let $c = 1 - 2^{-e+3}$. For any i, we have

$$h_{i,s+1}(x) = \begin{cases} \frac{(c-1)h_{i,s}(a)}{z-a}(x-a) + h_{i,s}(a), & \text{if } a \leq x \leq z, \\ \frac{(1-c)h_{i,s}(a)}{b-z}(x-b) + h_{i,s}(a), & \text{if } z < x < b. \end{cases}$$

Fix i,j. We argue that $d(h_{i,s}, h_{j,s}) = d(h_{i,s+1}, h_{j,s+1})$. Suppose that $h_{i,s}(a) \geq h_{j,s}(a)$. It is straightforward to check that for every $x \in J$, we have $0 \leq h_{i,s+1}(x) - h_{j,s+1}(x) \leq h_{i,s}(a) - h_{j,s}(a)$. Hence distances are preserved. Next we fix i,j,k and

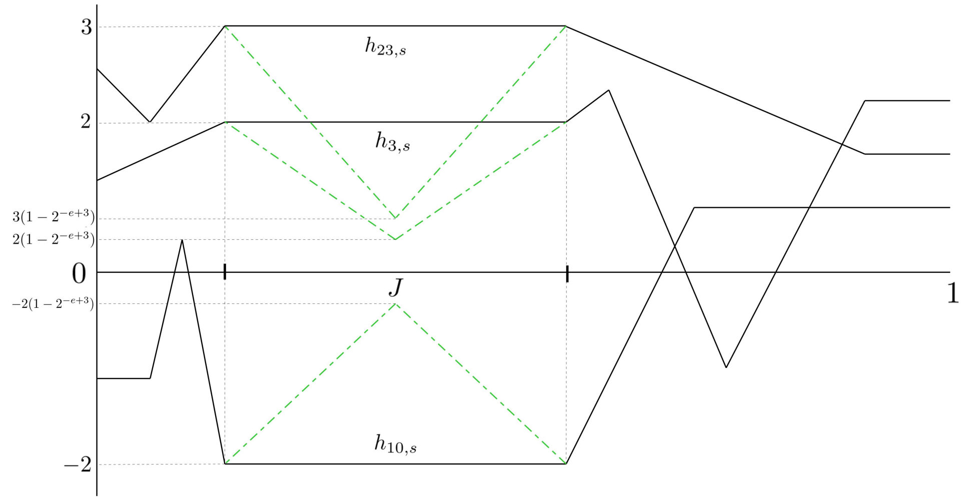

FIGURE 4. The dashed lines indicate the modifications needed to obtain $h_{i,s+1}$.

assume that $h_{i,s}(a) + h_{j,s}(a) = h_{k,s}(a)$. It is routine to check that for any $x \in J$ we have $h_{i,s+1}(x) + h_{j,s+1}(x) = h_{k,s+1}(x)$. □

4.4. **Construction.** At stage $s = \langle p, q, r\rangle$, let the first s many N-strategies act according to their instructions. Next, if q is odd we set $q' = p$ and if q is even we set q' to be a number such that $\widehat{l}_{q'} = h_{r_0,s} + h_{r_1,s}$, where $r = \langle r_0, r_1\rangle$. Now for this q' we see if $\widehat{l}_{q'}$ is among $(h_{i,s})_{i\le s}$. If it is not, we pick n least such that h_n has no approximation so far and set $h_{n,s} = \widehat{l}_{q'}$.

Declare distances and $+$ on the finite set $\{h_i\}_{i\le n}$ accordingly; that is, if we see $h_{i,s} + h_{j,s} = h_{k,s}$ for some $i, j, k \le n$, we declare that $h_i +^Y h_j = h_k$ if such a definition does not already exist in the structure Y. This ends the construction.

4.5. **Verification.**

Lemma 4.9. *$Y = (h_i)_{i\in\mathbb{N}}$ is a computable structure.*

Proof. For any $i \neq j$, the distance $d(h_i, h_j)$ is declared at the first stage s where both $h_{i,s}$ and $h_{j,s}$ are defined, and it is nonzero because, by construction, $h_{i,s} \neq h_{j,s}$. Now consider a stage s where both $h_{i,s}$ and $h_{j,s}$ are defined, and let p be such that $\widehat{l}_p = h_{i,s} + h_{j,s}$ (this p exists by Lemma 4.6(ii)). Now by the construction $h_i + h_j$ must receive a definition at or before stage $\langle p, 2s, \langle i, j\rangle\rangle$. □

Lemma 4.10. *The global requirements are satisfied.*

Proof. For each i let $M_i = \|h_{i,t_0}\|$ where t_0 is the first stage where h_i is given an interpretation. Requirement N_e acts at most once during the construction, and if requirement N_e acts at some stage s then for each i, $d(h_{i,s}, h_{i,s+1}) \le 2^{-e+3}M_i$. Now it follows that for each e there is a stage s_e of the construction such that for every $t > t' \ge s_e$, and every i, we have $d(h_{i,t}, h_{i,t'}) < 2^{-e}M_i$.

This means that global requirement (1) is satisfied, since the sequence $\{h_{i,t}\}_{t\in\mathbb{N}}$ is a Cauchy sequence in $cl(\hat{L})$. Global requirements (2) and (3) follow from Lemma

4.8. For (4) we fix m, e. Let e' be large enough so that $2^{-e'}\|\widehat{l}_m\| < 2^{-e}$, and consider a stage of the form $s' = \langle m, 2q+1, r\rangle > s_{e'}$ for some q, r. The construction at this stage ensures that $\widehat{l}_m = h_{k,s'}$ for some k. Then $d(\lim_s h_{k,s}, \widehat{l}_m) = d(\lim_s h_{k,s}, h_{k,s'}) \leq 2^{-e'}M_k < 2^{-e}$. □

Lemma 4.11. *The requirement N_e is satisfied.*

Proof. Clearly if N_e never acts then it is satisfied, so we assume that N_e acts at a stage s, when it sees that $d(h_{x,s}, \mathbf{1}) \leq 2^{-e+1}$, and $\theta_{e,s}(0) > e$. The requirement then proceeds to adjust $h_{x,s+1}$ on interval J. We have $h_{x,s}(a) \leq 1 + 2^{-e+1}$, and so $h_{x,s+1}(z) = (1 - 2^{-e+3})h_{x,s}(a) < 1 - 2^{-e+2}$. Since $h_{x,t}$ is never again modified in the interval J, we have $d(h_x, \mathbf{1}) > 2^{-e+2}$. Since $\theta_{e,s}(0) > e$ we have $d(\lim_t h_{\Theta_{e,t}(0)}, h_x) \leq 2^{-e+1}$ (in the closures of both structures Y and $\widehat{L}$, since they are isometric). Thus in $cl(\widehat{L})$, we must have $d(\lim_t h_{\Theta_{e,t}(0)}, \mathbf{1}) > 2^{-e+1}$ and so we cannot have $\lim_t h_{\Theta_{e,t}(0)} = \mathbf{1}$ in $cl(Y)$. □

Notice that the convergence of the interpretations $(h_{i,s})_{s\in\mathbb{N}}$ is not necessarily effectively rapid. For instance, we can assign Cauchy names to the points $\mathbf{1}$ and $-\mathbf{1}$ only with the help of the Halting problem. This concludes the proof of Theorem 4.2.

5. $C[0,1]$ is not computably categorical as a Banach algebra

Recall that the signature of Banach algebras include the symbol for the metric, the usual vector space operations, the multiplication symbol, and symbols for the additive identity $\mathbf{0}$ and the multiplicative identity $\mathbf{1}$.

By Fact 2.15, we can replace the usual Banach algebra signature by the simple one including only $+$ and $\times$ (and the metric). Suppose we wish to show that $C[0,1]$ is computably categorical in the language of Banach algebras. Then, given any computable structure Y and the nice structure L on $C[0,1]$, we have to be able to effectively and uniformly map each special point of L to a computable point of $cl(Y)$. However, the trouble with doing so is that, even if we know all the operations in the signature of Banach algebras are computable w.r.t. Y, it is still not clear how we can effectively approximate, say, an isomorphic image of the rational polynomial $x^3 - 1/2$ with respect to Y.

However, if we add a distinguished element representing the monomial $f(x) = x$ to the signature, then we can simply write down images of the polynomials over Q using the symbols of our signature. Thus, we have:

Fact 5.1. *The space $C[0,1]$ is computably categorical in the language of Banach algebras with an extra distinguished symbol for the function $f(x) = x$.*

Proof. Given the nice computable structure $\mathcal{P}$ of all rational-valued polynomials, and a computable structure Y on $C[0,1]$, we can define an isometry $\phi : \mathcal{P} \mapsto Y$ mapping $\sum_n r_n z^n \in \mathcal{P}$ to $\sum_n r_n \phi(x)^n \in cl(Y)$. The image $\phi(\sum_n r_n z^n)$ can be uniformly rapidly approximated w.r.t. Y. □

In fact, we will show that the space $C[0,1]$ is *not* computably categorical in the language of Banach algebras, by constructing a computable structure Z in which, roughly speaking, the "identity function $f(x) = x$" is a non-computable point of Z.

We have already mentioned the description of automorphisms of $(C[0,1], d, +, \times)$. As a consequence of this description, the automorphism orbit of the identity function $f(x) = x$ contains exactly homeomorphisms of $[0,1]$ onto itself. (The image of $f(x)$ is $\delta(x)f(g(x)) = \delta(x)g(x)$, where $g(x)$ is a homeomorphism of $[0,1]$ onto itself, but because $\mathbf{1}$ is in the signature and $\mathbf{1}(x) = \delta(x)\mathbf{1}(g(x)) = \delta(x)$, we can omit δ). In other words, it contains exactly the strictly monotonic continuous functions of norm 1 and equal to zero at one of the end-points.

Definition 5.2. Let $\mathcal{Q} = (q_i)_{i\in\mathbb{N}}$ be a dense effective sequence of continuous piecewise polynomial functions with rational coefficients (abbreviated p.p. functions) with finitely many rational breakpoints, without repetitions. That is, each $q_i \in C[0,1]$ is defined piecewise on intervals $I_1, \cdots, I_{n_i}$ where $[0,1] = I_1 \cup \cdots \cup I_{n_i}$ and for each interval I_m, $q_i = p_i^m$ on I_m, where $(p_i^m)_{i,m}$ is a computable collection of rational-valued polynomials.

It is clear that $\mathcal{Q}$ is a computable structure on $C[0,1]$ in the language of Banach spaces. Furthermore the set $\mathcal{Q} = (q_i)_{i\in\mathbb{N}}$ is effectively closed under addition and multiplication (unlike the structures L and $\widehat{L}$, where multiplication is only computable in the limit structure $cl(L)$ and $cl(\widehat{L})$).

It follows from the theorem below that the difficulty discussed before Fact 5.1 cannot be circumvented, and adding a constant symbol for the identity function is necessary in Fact 5.1.

Theorem 5.3. *There is computable structure Z on $(C[0,1], +, \times)$ and an onto isometry $\phi : cl(Z) \mapsto cl(\mathcal{Q})$ (preserving all Banach algebra operations) such that given any strictly monotonic function $f \in C[0,1]$, $\phi^{-1}(f)$ is not a computable point with respect to Z.*

In contrast to all our previous results, the map ϕ in the above theorem is not simply the identity map. In fact, $cl(Z) = C[0,\alpha]$ for some positive left-c.e real α, and ϕ is the Banach algebra isomorphism induced by a certain homeomorphism of $[0,\alpha]$ onto $[0,1]$. In the proof, we will be constructing a computable structure on $(C[0,\alpha], +, \times)$ which, of course, will correspond to a computable structure on $(C[0,1], +, \times)$ via the map ϕ. (More specifically, consider the collection of all ϕ-images of special points of the structure.) We obtain the following important corollary:

Corollary 5.4. *The space $C[0,1]$ is not computably categorical as a Banach algebra.*

Proof. The computable structure Z from the preceding theorem can not be computably isometric to $\mathcal{Q}$. Suppose $\psi : cl(\mathcal{Q}) \mapsto cl(Z)$ was a computable isometry. Let $f(x) = x$ be the identity function. Then $\phi\psi f$ would have to be a strictly monotonic function. Since f is a special point of $\mathcal{Q}$ and ψ is a computable isometry, this means that ψf is a computable point of Z, contradicting Theorem 5.3. □

The rest of this section is devoted to the proof of Theorem 5.3.

5.1. **Proof idea.** Notice that the vector space operations together with pointwise multiplication compute the constant function $\boldsymbol{r}$ for every $r \in Q$ (which outputs r at every point $x \in [0,1]$). Thus, there is a little hope to do any local vertical "distortion" of functions as we have done in Theorem 4.2 and Theorem 3.10, because any such strategy will cause $\phi^{-1}(\boldsymbol{r})$ to be a non-computable point of Z. Therefore we will have to use a strategy which is quite different from the previous arguments.

The key idea is in "going horizontal" instead. As usual we are building a computable structure $Z = (h_i)_{i\in\mathbb{N}}$ and maintaining a stage by stage interpretation $h_{i,s}$ of h_i in some nice structure. We wait for the e^{th} potential approximation to declare that it is an isometric preimage of a monotonic function with a good precision. The strategy will retarget $h_{i,s}$ so that this e^{th} potential approximation is incorrect. The reader might first try the following naive strategy: Pick some interval $[1-\delta, 1]$ and "reflect" $h_{i,s}$ on interval $[1-2\delta, 1-\delta]$ to the interval $[1-\delta, 1]$ for every interpretation introduced so far. That is, define

$$h_{i,s+1}(x) = \begin{cases} h_{i,s}(x), & \text{if } x < 1-\delta, \\ h_{i,s}(2-2\delta-x), & \text{if } x \geq 1-\delta, \end{cases}$$

which is illustrated in Figure 5.

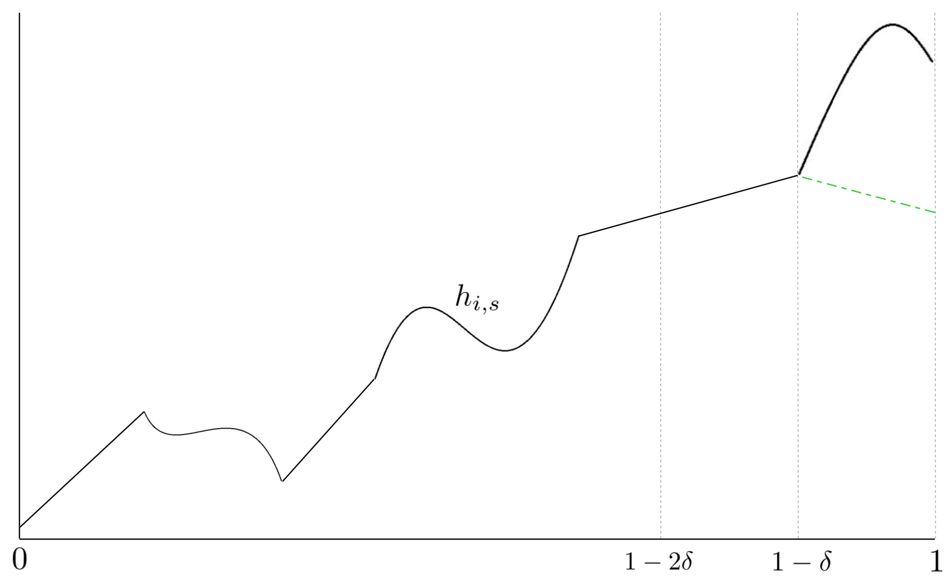

FIGURE 5. The naive strategy: The dashed lines indicate the modifications needed to get $h_{i,s+1}$.

This naive strategy clearly kills off the e^{th} potential approximation, since h_{s+1} does not look close to any strictly monotonic function, and will preserve the computability of constant functions. In fact, this strategy preserves at each stage, the operations $+$ and $\times$. Unfortunately it does not preserve distances, in the sense that $d(h_{i,s}, h_{j,s})$ need not be the same as $d(h_{i,s+1}, h_{j,s+1})$.

We modify this naive strategy as follows. Notice that for each real $\alpha \geq 1$, the space $C[0,\alpha]$ is isometric to $C[0,1]$ via the natural map which stretches or compresses the x-axis, i.e. the map ϕ_α that maps each $f \in C[0,\alpha]$ to $f(x\alpha) \in C[0,1]$, and which clearly preserves all pointwise properties of functions. So we *could* construct Z to be a subspace of $C[0,\alpha]$ instead of $C[0,1]$. To wit, we would allow the diagonalization strategy to enlarge the interval by increasing $\alpha_{s+1} > \alpha_s$, and reflecting the graph of $h_{i,s}$ on the interval $[2\alpha_s - \alpha_{s+1}, \alpha_s]$ to the interval

$[\alpha_s, \alpha_{s+1}]$:

$$h_{i,s+1}(x) = \begin{cases} h_{i,s}(x), & \text{if } x \leq \alpha_s, \\ h_{i,s}(\alpha_s - z), & \text{if } x = \alpha_s + z \text{ for } z > 0. \end{cases}$$

This is illustrated in Figure 6.

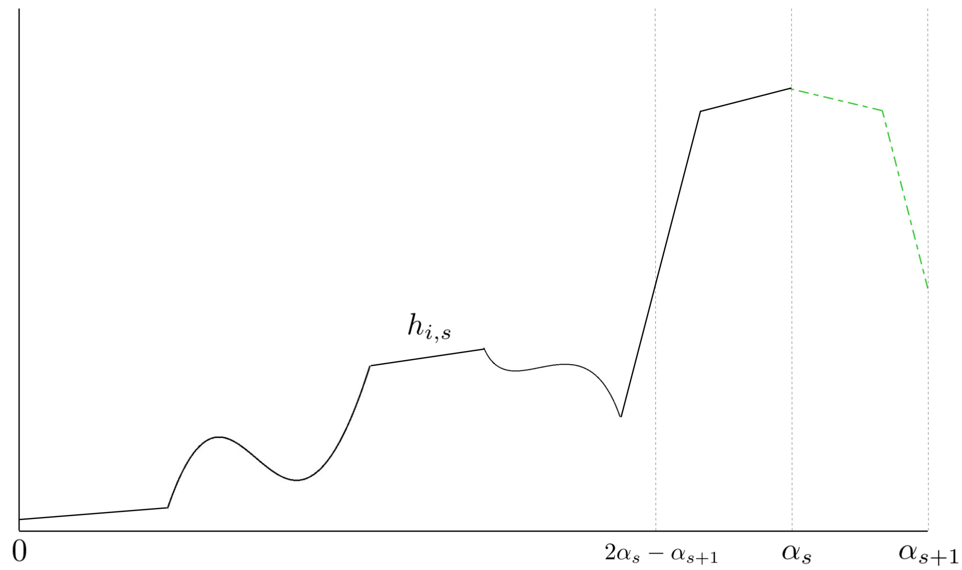

FIGURE 6. The main strategy: The dashed lines indicate the modifications needed to get $h_{i,s+1}$.

This strategy now clearly preserves all Banach algebra operations, since all operations are evaluated pointwise, and it kills off the e^{th} potential approximation for a strictly monotonic function.

Since our construction has to be effective and we require the structure we build Z to be computable, we need to be somewhat careful with how we set up the construction. Given any (index for a) left c.e. increasing approximation $\{\alpha_s\}_{s\in\mathbb{N}}$ of a c.e. real $1 \leq \alpha \leq 2$ we can effectively obtain (an index for) the computable structure $\mathcal{Q}_\alpha = (q_i^\alpha)_{i\in\mathbb{N}}$ on $C[0, \alpha]$ which consists of all rational p.p. valued functions on $[0, \alpha]$ which are constant on some interval $[\alpha - \delta, \alpha]$ for some $\delta > 0$. This is because each time α_s increase we can begin enumerating rational p.p. valued functions with breakpoints $\alpha_{s-1} \leq r < \alpha_s$. Each function enumerated in the structure $\mathcal{Q}_\alpha$ is computably specified even though we never know the actual value of α to any reasonable precision, because each function is declared constant on the rightmost piece. Furthermore it is easy to see that distance $d(q_i^\alpha, q_j^\alpha)$ is computable, and that the set $(q_i^\alpha)_{i\in\mathbb{N}}$ is effectively closed under $+$ and $\times$.

In the formal construction we will build structure Z and an approximation $\{\alpha_s\}$ of the left c.e. real α. Let $\widehat{\mathcal{Q}} = \mathcal{Q}_\alpha$. For completeness we mention a technical issue here; the less recursion theoretic inclined reader may ignore the subsequent comment:

Remark 5.5. We could build $\{\alpha_s\}$ and $\widehat{\mathcal{Q}}$ simultaneously during the construction, because to compute distances, $+$ and $\times$ on elements of $\widehat{\mathcal{Q}} = \mathcal{Q}_\alpha$ seen at stage s will only require the value of α_s, and we never need to look ahead. Alternatively for a slicker approach we can build $\{\alpha_s\}$ and assume by the Recursion Theorem that we are given, during the construction, an index for $\widehat{\mathcal{Q}} = \mathcal{Q}_\alpha$.

In this proof we will take $\widehat{\mathcal{Q}}$ instead of $\mathcal{Q}$ as the nice structure. We build $Z = (h_i)_{i\in\mathbb{N}}$ and a stage by stage interpretation $h_{i,s} \in \widehat{\mathcal{Q}}$ of h_i. At the end we take $h_i = \lim_s h_{i,s}$.

Before we begin the formal proof, we mention that there will be several technical issues in the verification of the construction. The first difficulty is that for each h_i we will have infinitely many strategies enlarging the interval and retargetting h_i as described in Figure 6. Suppose the e^{th} strategy is allowed to enlarge the interval by ϵ_e. If we are not careful we may end up with $h_i(x)$ having no limit as $x \to \alpha$, for instance, if the total variation is too large. To get around this difficulty we have to choose $\sum_{j>e} \epsilon_j$ to be much smaller than ϵ_e, and argue using a careful analysis of the total possible variation. The second difficulty is to show that ϕ is a map onto the closure $cl(\widehat{\mathcal{Q}})$. We will again use the notion of variation to show that the construction makes $\phi(Y)$ dense in $cl(\widehat{\mathcal{Q}})$.

5.2. **Strategies.** As before, d stands for the supremum metric. We fix a computable sequence $(\epsilon_e)_{e\in\mathbb{N}}$ of positive rationals such that

$$2^{-e} > \sum_{j>e} \epsilon_j.$$

The strategy for each N_e (to be defined) will *act* at most once during the construction. Let $E = \{e : N_e \text{ acts at some stage}\}$ and $E_s = \{e : N_e \text{ acts at a stage } < s\}$. We set $\alpha = 1 + \sum_{e\in E} \epsilon_e$ and $\alpha_s = 1 + \sum_{e\in E_s} \epsilon_e$. The real $\alpha = \lim_s \alpha_s$ is left c.e..

We fix an enumeration of $\widehat{\mathcal{Q}} = (\widehat{q}_m)_{m\in\mathbb{N}}$. At stage s we may assume, by Remark 5.5 that the functions $\widehat{q}_m$ for $m \leq s$ all have breakpoints smaller than α_s and are constant on $[\alpha_s, \alpha]$. At every stage s we have $h_{i,s} \in \widehat{\mathcal{Q}}_s = (\widehat{q}_m)_{m\leq s}$, but $h_i = \lim_s h_{i,s}$ does not have to be an element of $\widehat{\mathcal{Q}}$. However we will ensure that at every i and s, $h_{i,s} \restriction_{[0,\alpha_s]} = h_{i,s+1} \restriction_{[0,\alpha_s]}$. That is, every modification made at stage s is on the interval $(\alpha_s, \alpha_{s+1}]$.

We have to meet the following global requirements:

(1) For every i, $h_i = \lim_s h_{i,s}$ exists in $cl(\widehat{\mathcal{Q}})$.
(2) For every i, j and s, we have $d(h_{i,s}, h_{j,s}) = d(h_{i,s+1}, h_{j,s+1})$.
(3) For every i, j, k and s, we have $h_{i,s} + h_{j,s} = h_{k,s} \Rightarrow h_{i,s+1} + h_{j,s+1} = h_{k,s+1}$.
(4) For every i, j, k and s, we have $h_{i,s} \times h_{j,s} = h_{k,s} \Rightarrow h_{i,s+1} \times h_{j,s+1} = h_{k,s+1}$.
(5) For each m and each e, there is some k such that $d(\lim_s h_{k,s}, \widehat{q}_m) \leq 2^{-e}$.

We will also explicitly make the structure closed under the operations $+$ and $\times$. These global requirements ensure that $cl(Z)$ and $cl(\widehat{\mathcal{Q}})$ are isometric via ϕ preserving $+$ and $\times$, since these operations are preserved at every stage. Hence $cl(Z)$ and $cl(\mathcal{Q})$ are isometric in the language of Banach algebras.

The key set of requirements to meet are:

N_e : $\Theta_e(0)$ is not a fast Cauchy name in Z for (the preimage of) a strictly monotonic function in $C[0,1]$.

Collectively the requirements N_e clearly imply that for any strictly monotonic function $f \in C[0,1]$, $(\phi_\alpha \phi)^{-1}(f)$ is not a computable point of Z. The strategy for N_e will ensure that $\lim_s h_{\Theta_{e,s}(0)}$ is not a strictly monotonic function in $C[0,\alpha]$. This ensures that N_e is met because ϕ_α is simply a scaling of the x-axis.

Strategy for N_e: Wait for a stage s such that for some $t \le s$, $\Theta_{e,t}(0)$ outputs a number n where

$$|\, h_{n,s}(\alpha_s - \epsilon_e) - h_{n,s}(\alpha_s)\,| > 2^{-\theta_{e,t}(0)+3}.$$

That is, at stage s we find that Θ outputs an index n (possibly at a past stage $t < s$) but which $h_{n,s}(\alpha_s - \epsilon_e)$ and $h_{n,s}(\alpha_s)$ now look sufficiently far apart. Note that $d(h_n, \lim_s h_{\Theta_{e,s}(0)}) < 2^{-\theta_{e,t}(0)}$.

If such a stage s is found, we set $\alpha_{s+1} = \alpha_s + \epsilon_e$ and for every interpretation $h_{i,s}$ introduced so far, redefine it as in Figure 6. That is, set

$$h_{i,s+1}(x) = \begin{cases} h_{i,s}(x), & \text{if } x \le \alpha_s, \\ h_{i,s}(\alpha_s - z), & \text{if } x = \alpha_s + z \text{ for } z \in (0, \epsilon_e]. \end{cases}$$

We say in this case that the strategy *acts.* This strategy will never have to act again. Notice that any modification to $h_{i,s}$ at stage s leave the values of $h_{i,s}(x)$ for $x \le \alpha_s$ untouched.

Lemma 5.6. *Distances and the operations $+$ and $\times$ are preserved under this action.*

Proof. Fix i, j. Since $h_{i,s}, h_{j,s}$ are constant on the interval $[\alpha_s, \alpha]$, we have

$$d(h_{i,s}, h_{j,s}) = \sup_{x \in [0,\alpha_s]} |h_{i,s}(x) - h_{j,s}(x)|.$$

Since $\sup_{x\in[\alpha_s,\alpha_{s+1}]} |h_{i,s+1}(x) - h_{j,s+1}(x)| \le \sup_{x \in [0,\alpha_s]} |h_{i,s}(x) - h_{j,s}(x)|$, we clearly have $d(h_{i,s+1}, h_{j,s+1}) = d(h_{i,s}, h_{j,s})$. It is also straightforward to check that the operations $+$ and $\times$ (being evaluated pointwise) are preserved under this action. □

5.3. **Construction.** At stage $s = \langle p, q, r\rangle$, let strategy N_e for the least $e < s$ which requires action act according to the strategy above.

Next, if $q \equiv 0 (\text{mod } 3)$ we set $m = p$ and if $q \equiv 1 (\text{mod } 3)$ we set m to be a number such that $\widehat{q}_m = h_{r_0,s} + h_{r_1,s}$, where $r = \langle r_0, r_1\rangle$. Finally if $q \equiv 2 (\text{mod } 3)$ we set m to be a number such that $\widehat{q}_m = h_{r_0,s} \times h_{r_1,s}$, where $r = \langle r_0, r_1\rangle$. Note that this m can be found at stage s, since the finite structure $\widehat{\mathcal{Q}}_s$ is effectively closed under $+$ and $\times$. Now for this m we see if $\widehat{q}_m$ is among $(h_{i,s})_{i \le s}$. If it is not, we pick n least such that h_n has no approximation so far and set $h_{n,s} = \widehat{q}_m$.

Declare distances, $+$ and $\times$ on the finite set $\{h_i\}_{i \le n}$ accordingly; that is, if we see $h_{i,s} + h_{j,s} = h_{k,s}$ for some $i, j, k \le n$, we declare that $h_i +^Y h_j = h_k$ if such a definition does not already exist in the structure Y. Similarly for $\times$. This ends the construction.

5.4. **Verification.**

Lemma 5.7. *$Z = (h_i)_{i \in \mathbb{N}}$ is a computable structure.*

Proof. This is proved the same way as Lemma 4.9. □

The tedious part of the verification lies in estimating, for each i and each s, the distance $d(h_{i,s}, h_{i,s+1})$. To do this we have to analyze carefully the actions taken during the construction. For each i we let s_i be the first stage such that

h_i is given an interpretation. Given a closed interval $J \subseteq [0, \alpha_{s_i}]$ we let $D(J, i) = \sup_{z \in J} h_{i,s_i}(z) - \inf_{z \in J} h_{i,s_i}(z)$. Since h_{i,s_i} is piecewise continuously differentiable (with only finitely many pieces), we let $C_i = \sup |h'_{i,s_i}(z)|$, where the supremum is taken over all points $z \in [0, \alpha_{s_i}]$ such z is not a breakpoint. It is easy to check, by the Mean Value Theorem, that:

Fact 5.8. *For any $J \subseteq [0, \alpha_{s_i}]$, we have $D(J, i) \leq C_i |J|$.*

Now we prove the following technical lemma about the construction.

Lemma 5.9. *Given any i, and any $s \geq s_i$, there exists a sequence of closed intervals $J_0^s, J_1^s, \cdots, J_{k_s}^s$ such that the following conditions hold:*

(i) *For every $j \leq k_s$, $J_j^s \subset [0, \alpha_{s_i}]$, and $\sum_{j \leq k_s} |J_j^s| = \alpha_s$.*
(ii) *For every $j \leq k_{s-1}$, we have $J_j^s = J_j^{s-1}$.*
(iii) *For each $j \leq k_s$, either*
  (1) *$h_{i,s} \restriction_{\widehat{J}_j^s} = h_{i,s_i} \restriction_{J_j^s}$, in the sense that $h_{i,s}(\min \widehat{J}_j^s + z) = h_{i,s_i}(\min J_j^s + z)$ for every $z \leq |J_j^s|$, or*
  (2) *$h_{i,s} \restriction_{\widehat{J}_j^s}$ is the mirror image of $h_{i,s_i} \restriction_{J_j^s}$, in the sense that $h_{i,s}(\min \widehat{J}_j^s + z) = h_{i,s_i}(\max J_j^s - z)$ for every $z \leq |J_j^s|$,*

  where $\widehat{J}_j^s = \left[\sum_{j' < j} |J_{j'}^s|,\ \sum_{j' \leq j} |J_{j'}^s|\right]$.

Proof. Fix i. This follows by a straightforward induction on $s \geq s_i$. For $s = s_i$ simply take $J_0^s = [0, \alpha_{s_i}]$. The induction step then follows easily because $h_{i,s+1}$ is obtained from $h_{i,s}$ by keeping $h_{i,s} \restriction_{[0,\alpha_s]}$ unchanged and then reflecting $h_{i,s} \restriction_{[2\alpha_s - \alpha_{s+1}]}$ over to $[\alpha_s, \alpha_{s+1}]$.

Specifically, assume that at stage s we have the sequence $J_0^s, \cdots, J_{k_s}^s$. Suppose the construction at s increases α_s to α_{s+1}, where $2\alpha_s - \alpha_{s+1} \in \widehat{J}_{j_0}^s$ for some $j_0 \leq k_s$. We may assume that $2\alpha_s - \alpha_{s+1}$ is not an endpoint of $\widehat{J}_{j_0}^s$, otherwise it is easy. Now let $k_{s+1} = k_s + (k_s - j_0 + 1)$. Let $J_j^{s+1} = J_j^s$ for every $j \leq k_s$ and $J_{k_s+j+1}^{s+1} = J_{k_s-j}^s$ for each $j < k_s - j_0$. Finally for $J_{k_{s+1}}^{s+1}$ we will set equal to the right subinterval of $J_{j_0}^s$ of the appropriate length ($= \max \widehat{J}_{j_0}^s - (2\alpha_s - \alpha_{s+1})$) if $h_{i,s} \restriction_{\widehat{J}_{j_0}^s} = h_{i,s_i} \restriction_{J_{j_0}^s}$. On the other hand if $h_{i,s} \restriction_{\widehat{J}_{j_0}^s}$ is the mirror image of $h_{i,s_i} \restriction_{J_{j_0}^s}$ then we set $J_{2k_s-j_0+1}^{s+1}$ equal to the left subinterval of $J_{j_0}^s$ of the same length ($= \max \widehat{J}_{j_0}^s - (2\alpha_s - \alpha_{s+1})$). □

Lemma 5.10. *Suppose N_e acts at some stage s. Then for each i where $s_i \leq s$, $d(h_{i,s}, h_{i,s+1}) \leq \epsilon_e C_i$.*

Proof. Let $z_j = \min \hat{J}_j^{s+1}$. Since the functions are continuous, let $\tilde{z} \geq \alpha_s$ be the point where $d(h_{i,s}, h_{i,s+1}) = |h_{i,s+1}(z_{k_s+1}) - h_{i,s+1}(\tilde{z})|$. Hence it follows that $d(h_{i,s}, h_{i,s+1}) \leq |h_{i,s+1}(z_{k_s+1}) - h_{i,s+1}(z_{k_s+2})| + |h_{i,s+1}(z_{k_s+2}) - h_{i,s+1}(z_{k_s+3})| + \cdots + |h_{i,s+1}(z_{k_s+p}) - h_{i,s+1}(\tilde{z})|$, where $\tilde{z} \in \hat{J}_p^{s+1}$. Now by Lemma 5.9 this is bounded by

$$\sum_{k_s < r \leq k_{s+1}} D(J_r^{s+1}, i).$$

By Fact 5.8 this is bounded by

$$\sum_{k_s < r \leq k_{s+1}} C_i |J_r^{s+1}| = C_i \epsilon_e.$$

□

Lemma 5.10 provides the necessary upperbound to proceed with the rest of the verification.

Lemma 5.11. *The global requirements are satisfied.*

Proof. Global requirement (2), (3) and (4) follow from Lemma 5.6. For (1) fix i. For every e there is a stage s' after which no requirement $N_{e'}$ for $e' \leq e$ acts. Then for every stage $t > t' \geq s'$, we have by Lemma 5.10, that $d(h_{i,t}, h_{i,t'}) < 2^{-e}C_i$. For (5) we now proceed as in Lemma 4.10. □

Lemma 5.12. *The requirement N_e is satisfied.*

Proof. Fix e. Assume for a contradiction that $\left(h_{\Theta_{e,s}(0)}\right)_{s\in\mathbb{N}}$ is a fast converging sequence which converges to a strictly monotonic function H in $C[0,\alpha]$. Since H is continuous, let e' be a number large enough so that for all large enough stages s, we have $|H(\alpha_s) - H(\alpha_s - \epsilon_e)| > 2^{-e'}$. Fix t so that $\Theta_{e,t}(0)$ outputs number n such that $\theta_{e,t}(0) > e' + 5$, and such that $|h_n(\alpha_s) - h_n(\alpha_s - \epsilon_e)| > 2^{-e'-1}$ for almost every stage $s > t$. Now it is easy to see that at almost every stage $s > t$, N_e will satisfy the conditions for it to act.

Now let N_e act at stage s. Let $t < s$ be the stage where $\Theta_{e,t}(0)$ outputs number n satisfying $|h_{n,s}(\alpha_s - \epsilon_e) - h_{n,s}(\alpha_s)\,| > 2^{-\theta_{e,t}(0)+3}$. Recall that $d(h_n, H) < 2^{-\theta_{e,t}(0)}$. Without loss of generality assume that $h_{n,s}(\alpha_s - \epsilon_e) < h_{n,s}(\alpha_s)$. The action at stage s ensures that $h_n(\alpha_s - \epsilon_e) + 2^{-\theta_{e,t}(0)+3} < h_n(\alpha_s)$ and $h_n(\alpha_s + \epsilon_e) + 2^{-\theta_{e,t}(0)+3} < h_n(\alpha_s)$.

Now examining the values of $H(\alpha_s - \epsilon_e), H(\alpha_s)$ and $H(\alpha_s + \epsilon_e)$ reveals a contradiction: $H(\alpha_s - \epsilon_e) < H(\alpha_s)$ and $H(\alpha_s + \epsilon_e) < H(\alpha_s)$. □

6. A short conclusion

As we have already mentioned in the preliminaries, we have not touched several combinations of symbols $+, (r\cdot)_{r\in Q}, \times, 1, 0$, as well us some other operations such as lattice operations, since not all of these combinations we find natural as signatures, and because we wished to keep the paper shorter. However, if we consider the relation "to effectively determine" on the family of signature symbols, we get a reduction. A pure theoretical curiosity could lead us to the further study of this reduction.

There are lots of problems related to our results which have not been touched so far. We could pick any other classical space such as $L_3[0,1]$ and ask the same questions we were addressing in our paper. Also, it is not clear if there is a Banach space which is computably categorical, but whose associated metric space is not computably categorical. Similarly, can we find a Banach algebra which is computably categorical, but so that the associated Banach space is not? What can be said about the computable dimension of classical Banach spaces and algebras, including $C[0,1]$? Can we find a Banach space of finite computable dimension $\neq 1$? We expect that new computability-analytic methods are needed to answer these and similar questions.

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